The Polytope of Pointed Pseudotriangulations, and Delone and anti-Delone Pseudotriangulations

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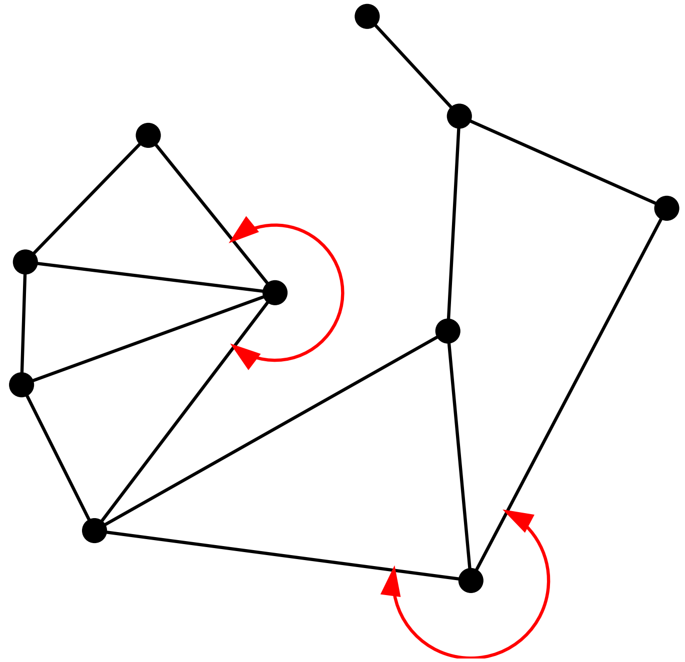
1. Pseudotriangulations: basic definitions and properties
2. The pointed pseudotriangulation polytope
3. Locally convex surfaces and lifted pseudotriangulations
4. Canonical pseudotriangulations

Pointed Vertices

A *pointed* vertex is incident to an angle $> 180^\circ$ (a *reflex* angle or *big* angle).



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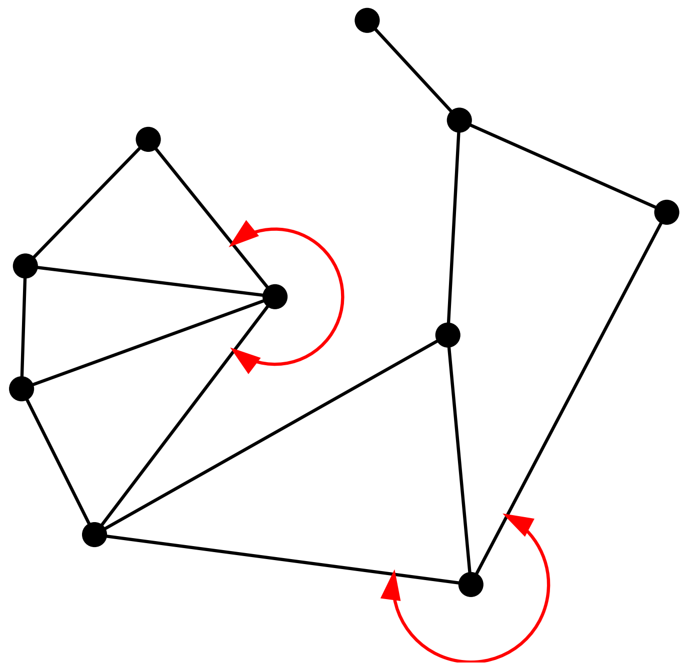

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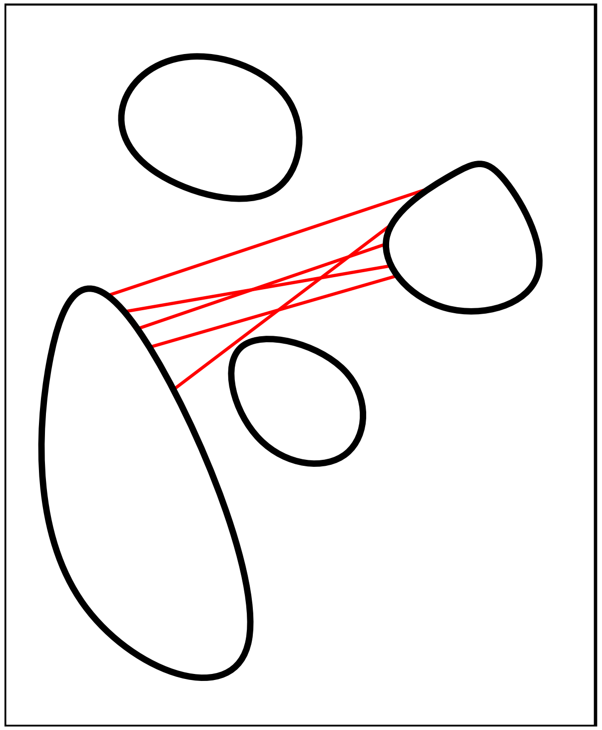
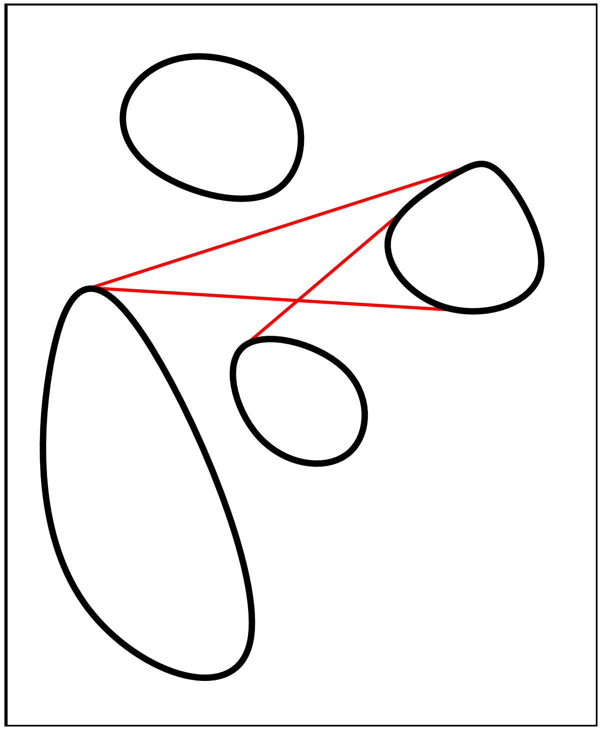
Where do pointed vertices arise?

Visibility among convex obstacles

Equivalence classes of *visibility segments*. Extreme segments are *bitangents* of convex obstacles.

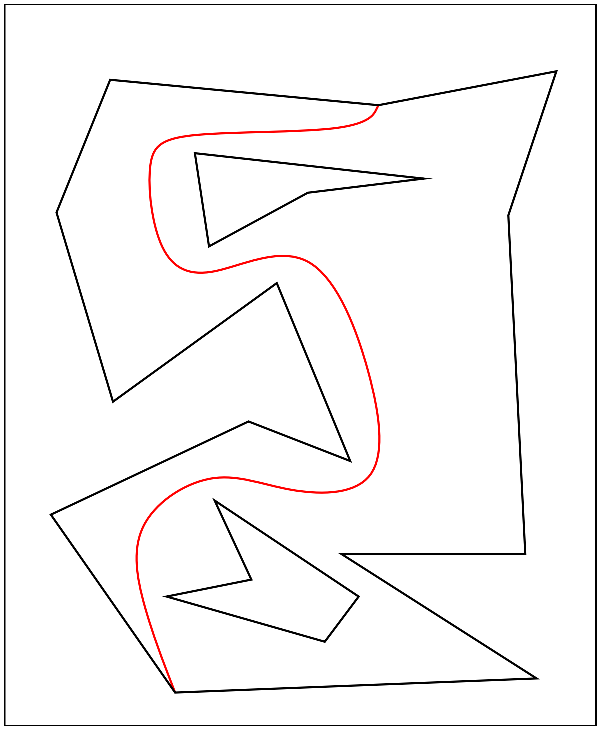


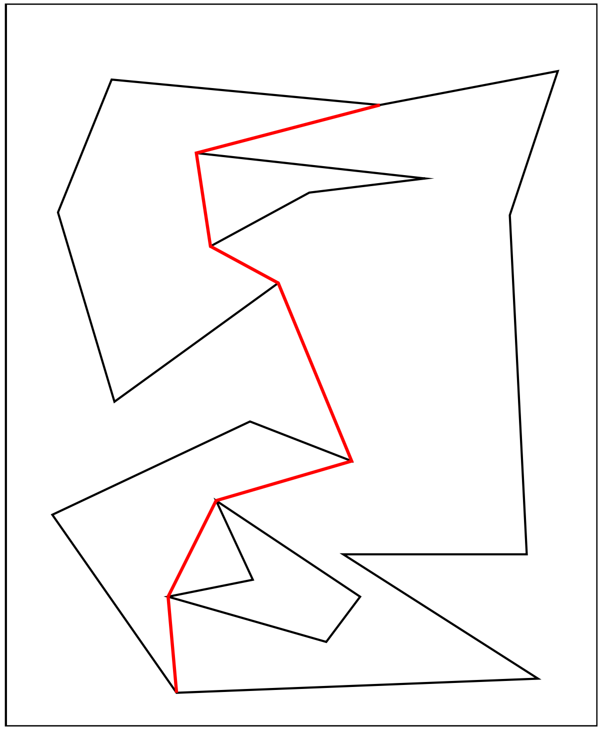


[Pocchiola and Vegter 1996]

Geodesic shortest paths

Shortest path (with given homotopy) turns only at pointed vertices. Addition of shortest path edges leaves intermediate vertices pointed.





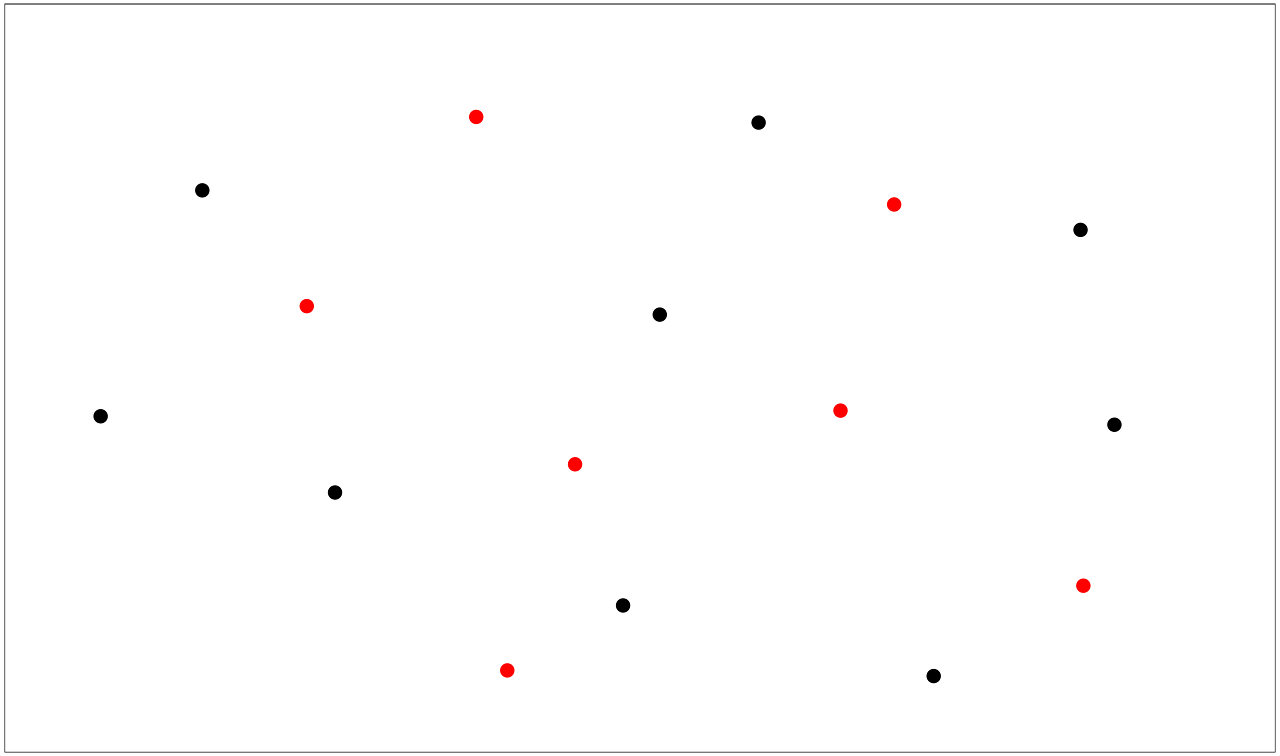
→ *geodesic* triangulations of a simple polygon

[Chazelle,Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink

Pseudotriangulations

Given: A set V of vertices, a subset $V_p \subseteq V$ of *pointed vertices*.

A *pseudotriangulation* is a maximal (with respect to $\subseteq$) set of non-crossing edges with all vertices in V_p pointed.



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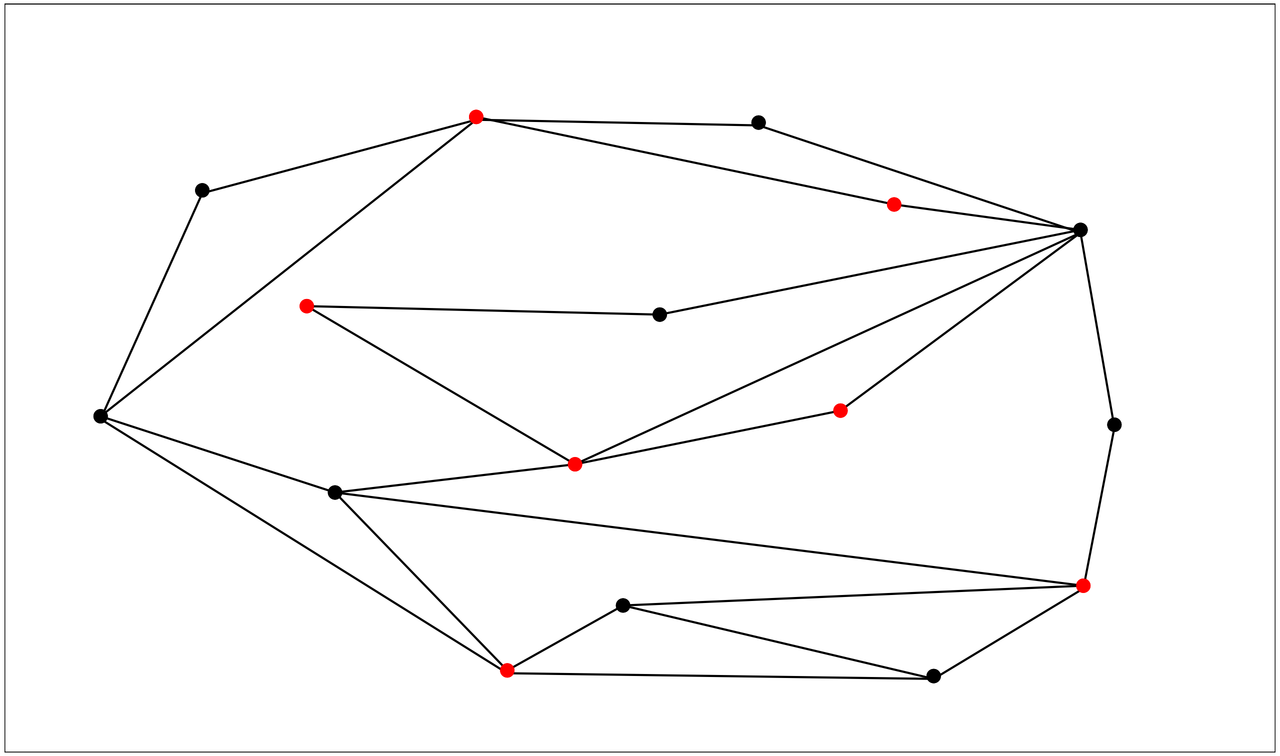

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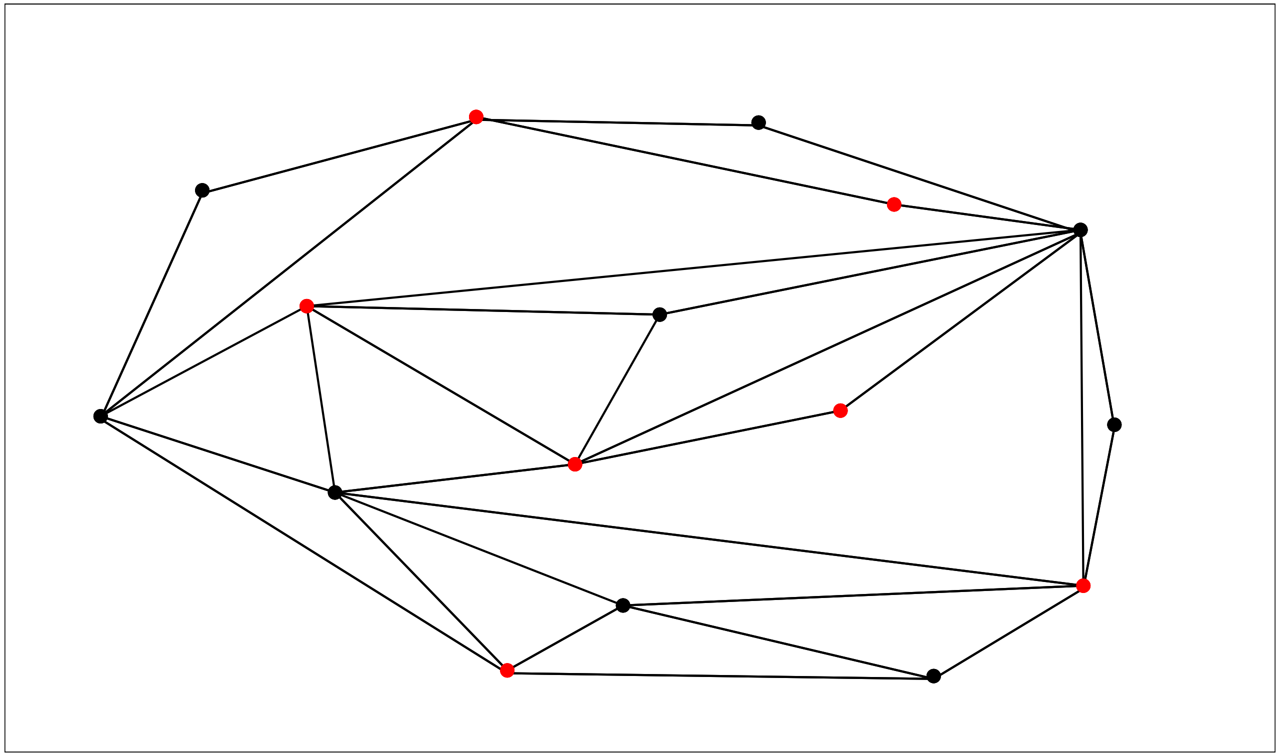

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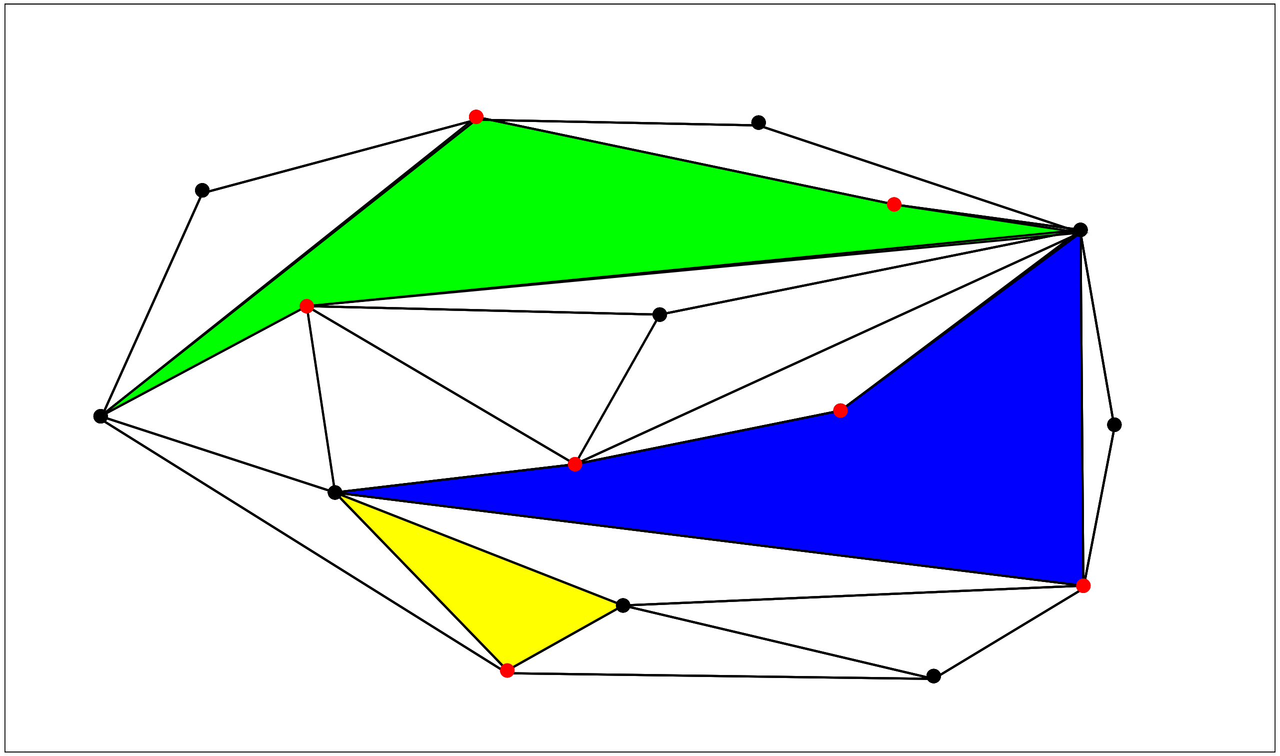
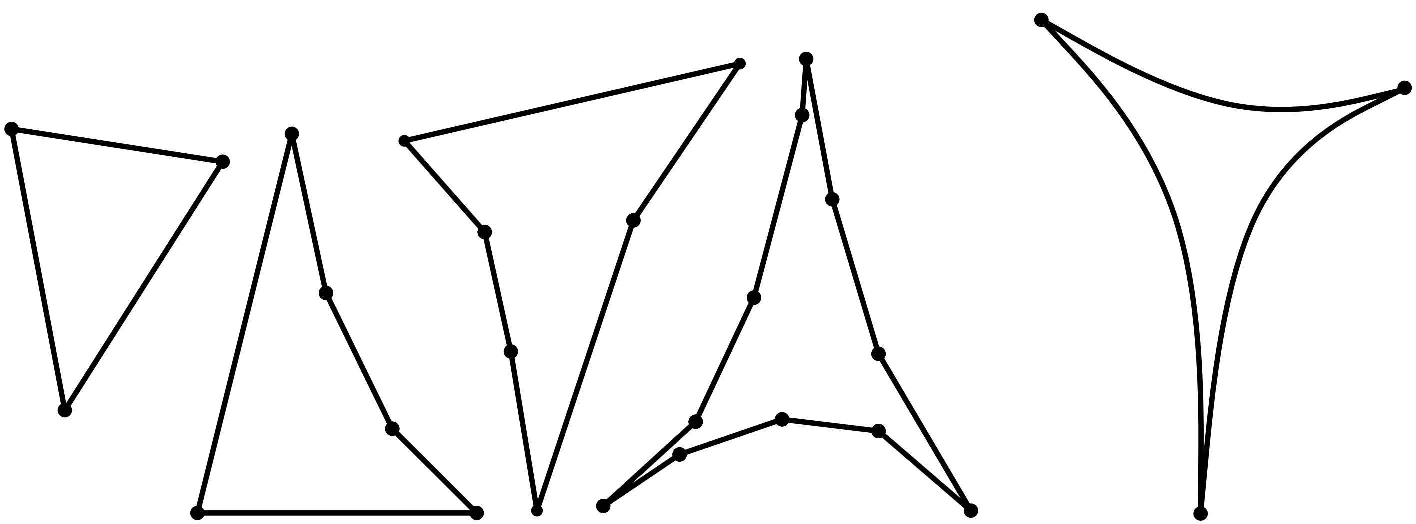


Pseudotriangles

A pseudotriangle has three convex *corners* and an arbitrary number of reflex vertices ($> 180^{\circ}$).



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Proof. $(1) \implies (2)$ All convex hull edges are in E.
$\longrightarrow$ decomposition of the polygon into faces.
Need to show: If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.

Characterization of pseudotriangulations

Lemma. *If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.*

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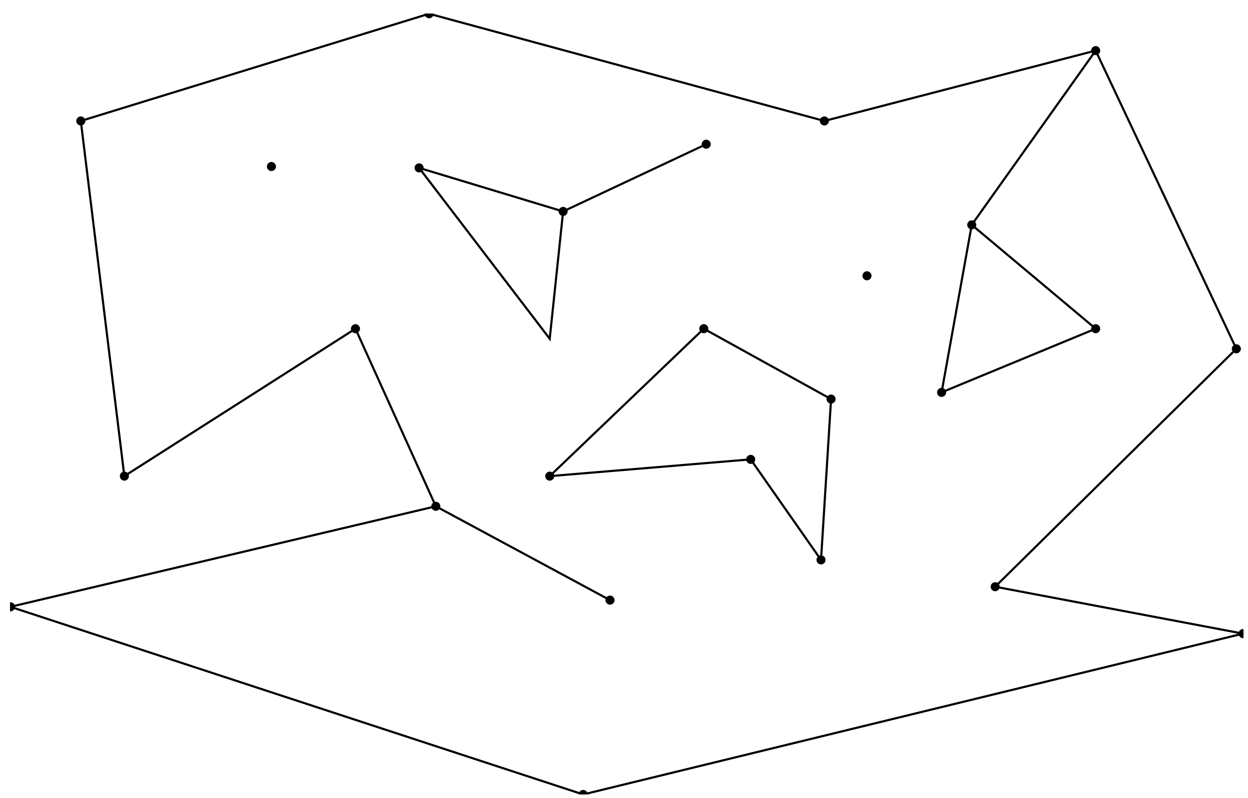
Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.



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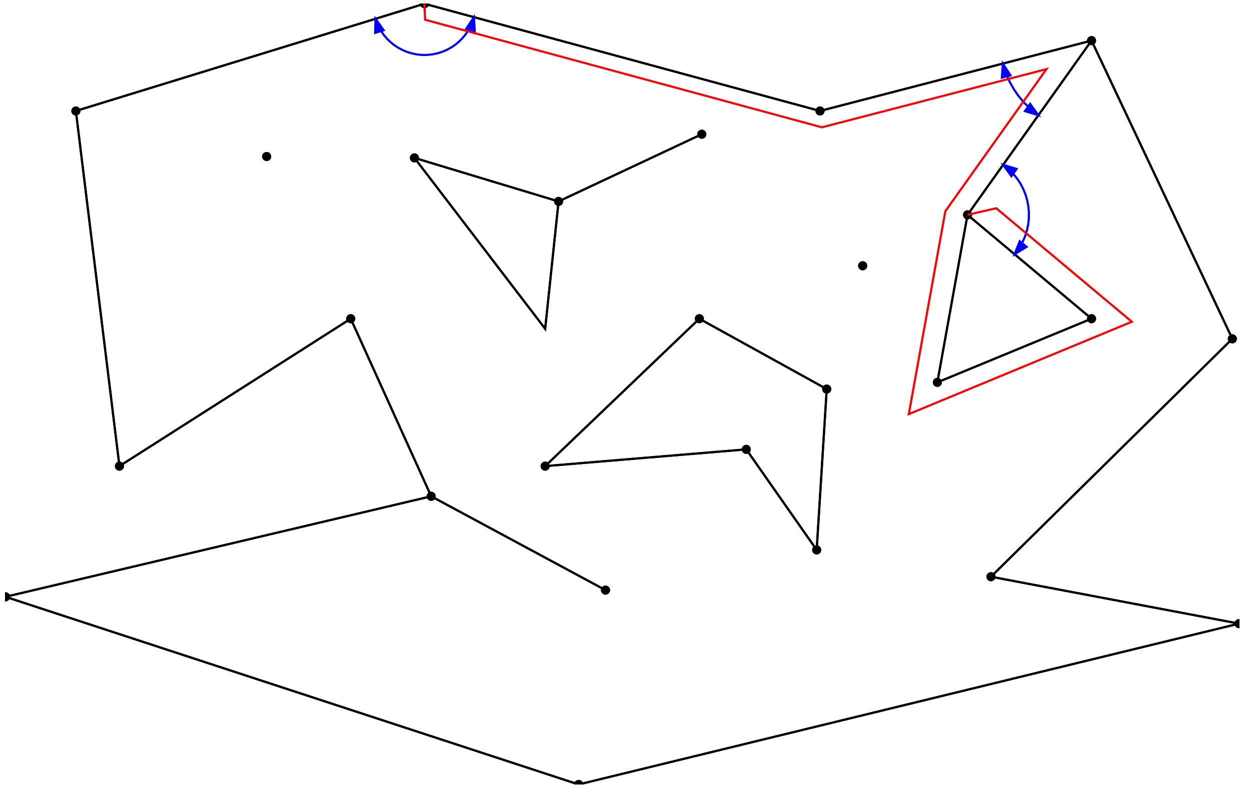

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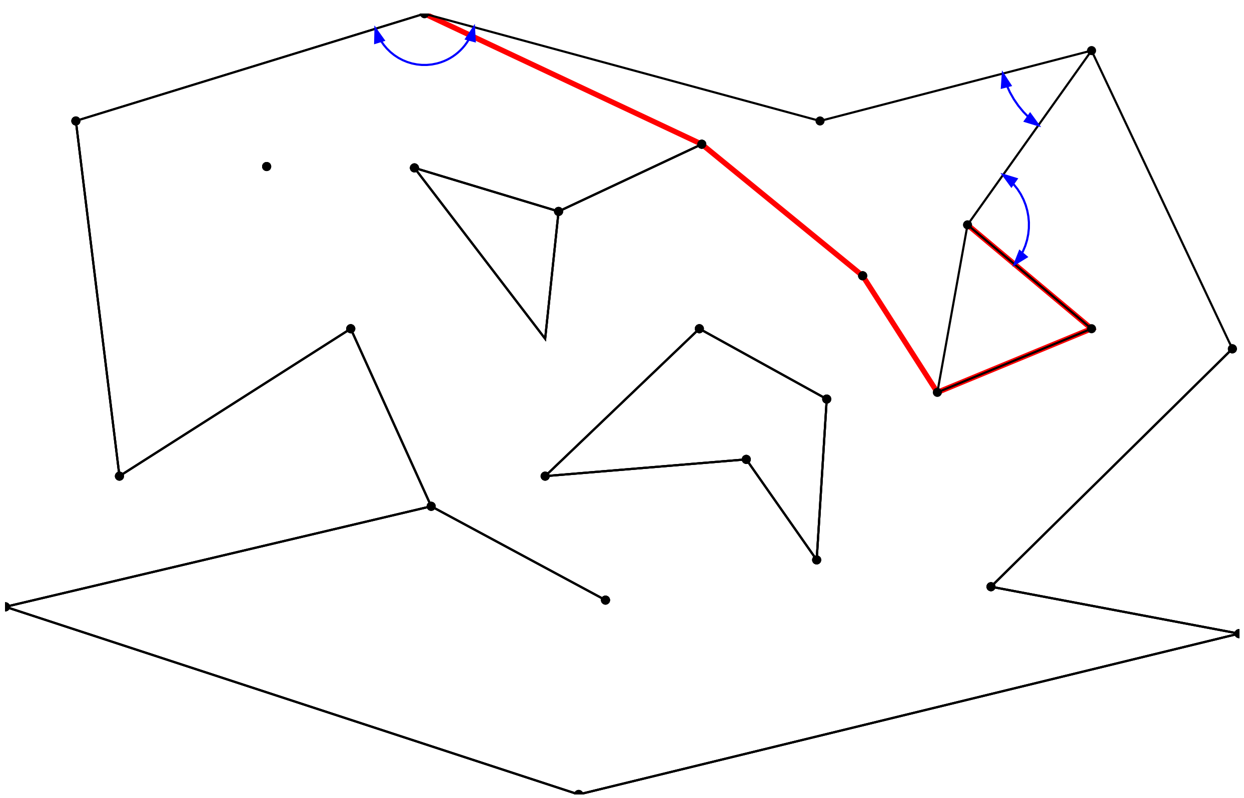
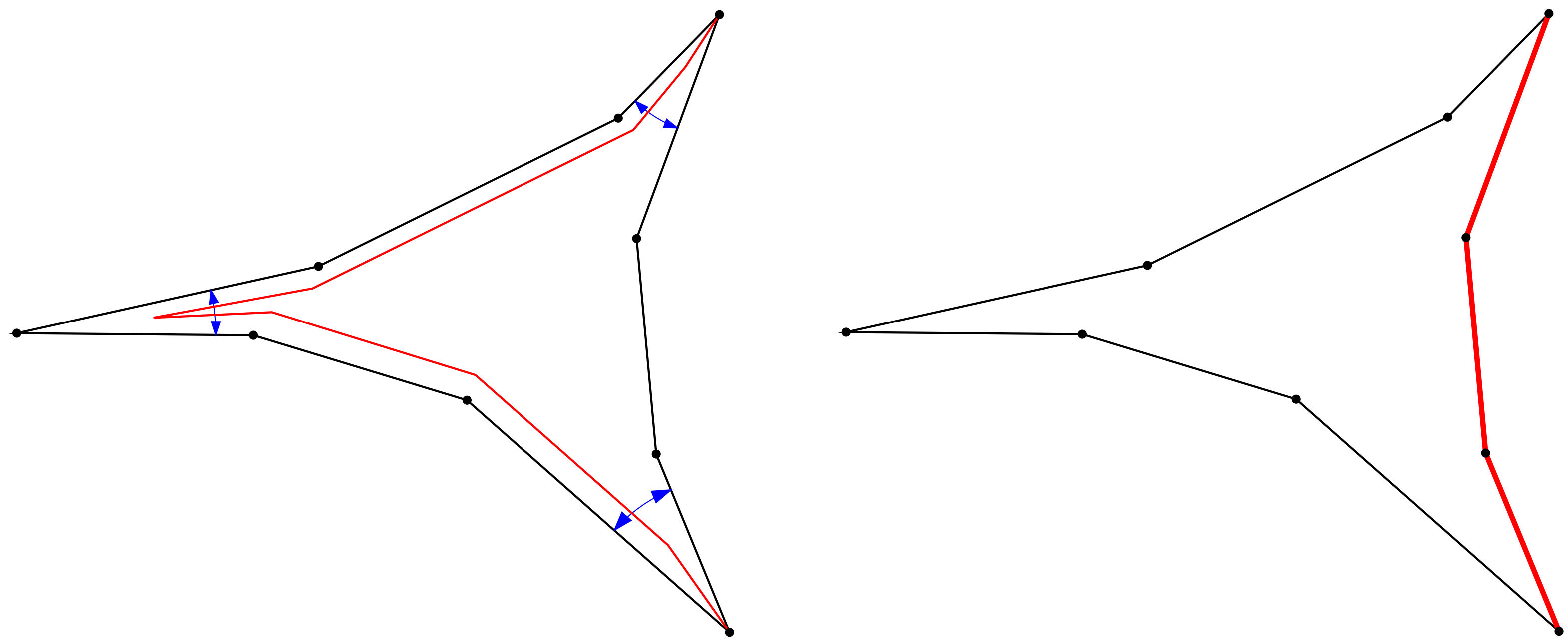


Characterization of pseudotriangulations, continued

A new edge is always added, unless the face is already a pseudotriangle (without inner obstacles).

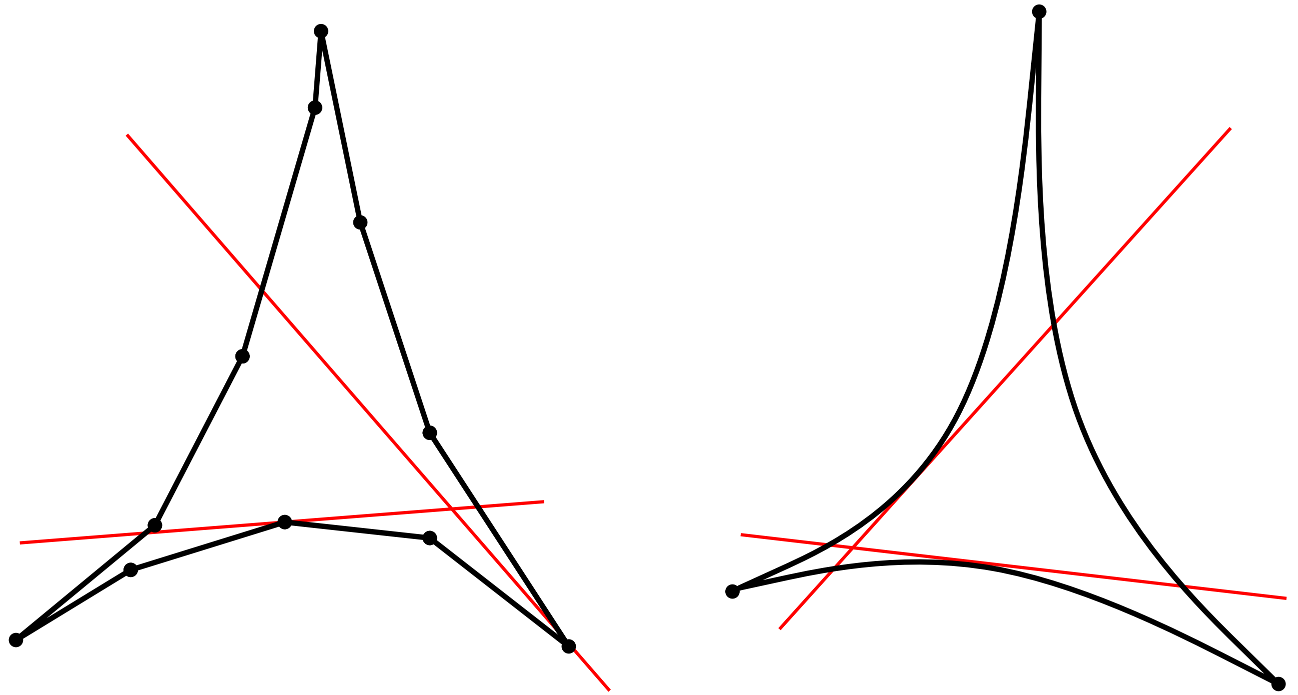


[Rote, C. A. Wang, L. Wang, Xu 2003]

Tangents of pseudotriangles

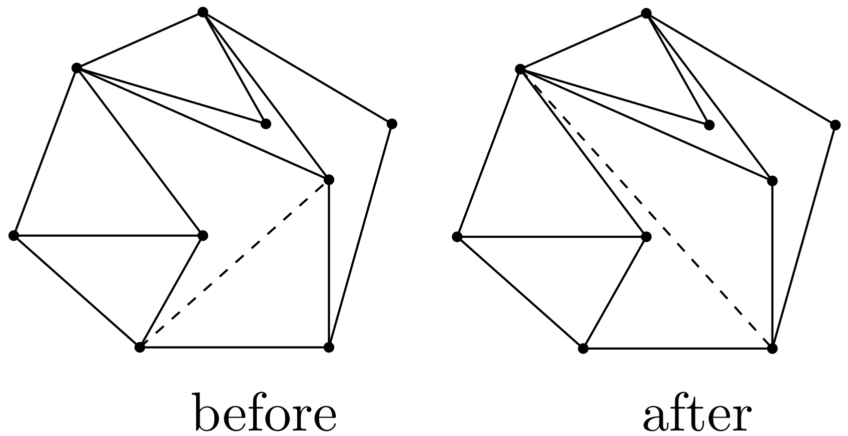
"Proof. $(2) \implies (1)$ No edge can be added inside a pseudotriangle without creating a nonpointed vertex."

For every direction, there is a unique *tangent line* which is "tangent" at a reflex vertex or "cuts through" a corner.



Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.



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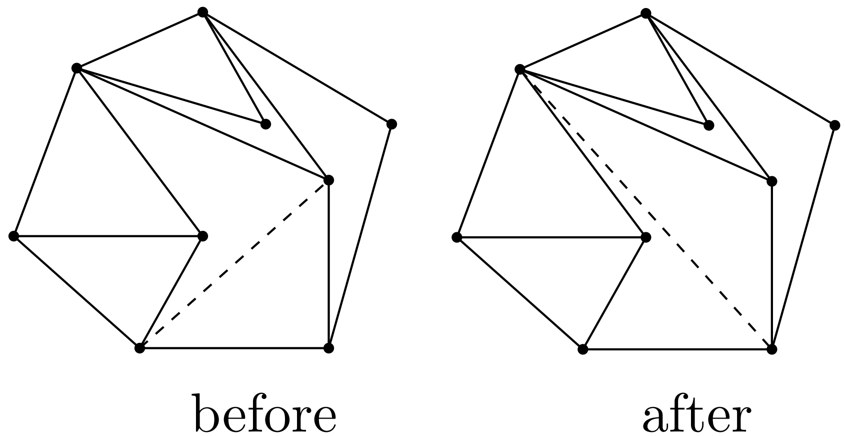
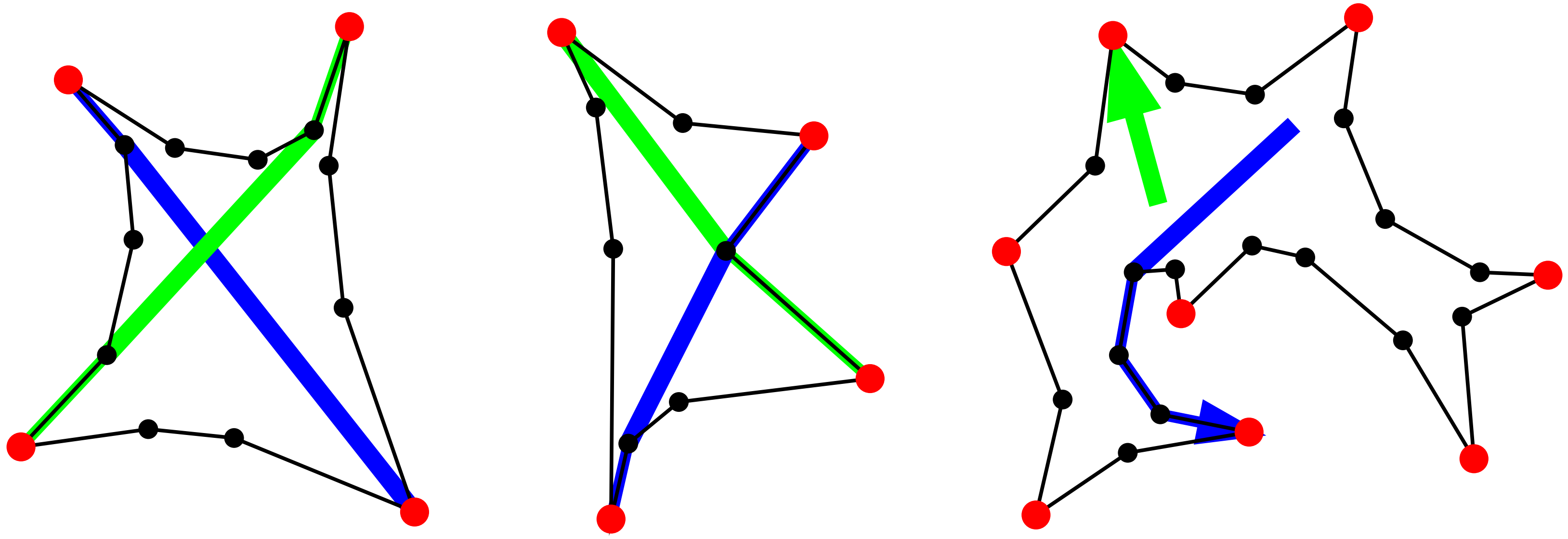


The flip graph is connected.
Its diameter is $O(n \log n)$. [Bespamyatnikh 2003]

Flipping

Every *tangent ray* can be continued to a geodesic path running along the boundary to a corner, in a unique way.

Every pseudoquadrangle has precisely two diagonals, which cut it into two pseudotriangles.



Vertex and face counts

Lemma. *A pseudotriangulation with* x *nonpointed and* y *pointed vertices has* $e = 3x + 2y - 3$ *edges and* $2x + y - 2$ *pseudotriangles.*

Corollary. *A pointed pseudotriangulation with* n *vertices has* $e = 2n - 3$ *edges and* $n - 2$ *pseudotriangles.*

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$$\underbrace{\sum_t k_t + k_{\text{outer}}}_{2e} - 3|T| = y$$

$$e + 2 = (|T| + 1) + (x + y) \qquad \text{(Euler)}$$

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Corollary. *A pointed graph with* $n \geq 2$ *vertices has at most* $2n - 3$ *edges.*

Pseudotriangulations/ Geodesic Triangulations

Applications:

- data structures for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, and Snoeyink 1994] and visibility [Pocchiola and Vegter 1996]
- kinetic collision detection [Agarwal, Basch, Erickson, Guibas, Hershberger, Zhang 1999–2001] [Kirkpatrick, Snoeyink, and Speckmann 2000] [Kirkpatrick & Speckmann 2002]
- art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]

2. A polyhedron for pointed pseudotriangulations

Theorem. *For every set S of points in general position, there is a convex $(2n-3)$-dimensional polyhedron X whose vertices correspond to the pointed pseudotriangulations of S.*

[Rote, Santos, Streinu 2003]

There is one inequality for each pair of points. At a vertex of X:

tight inequalities $\leftrightarrow$ edges of a pointed pseudotriangulation.

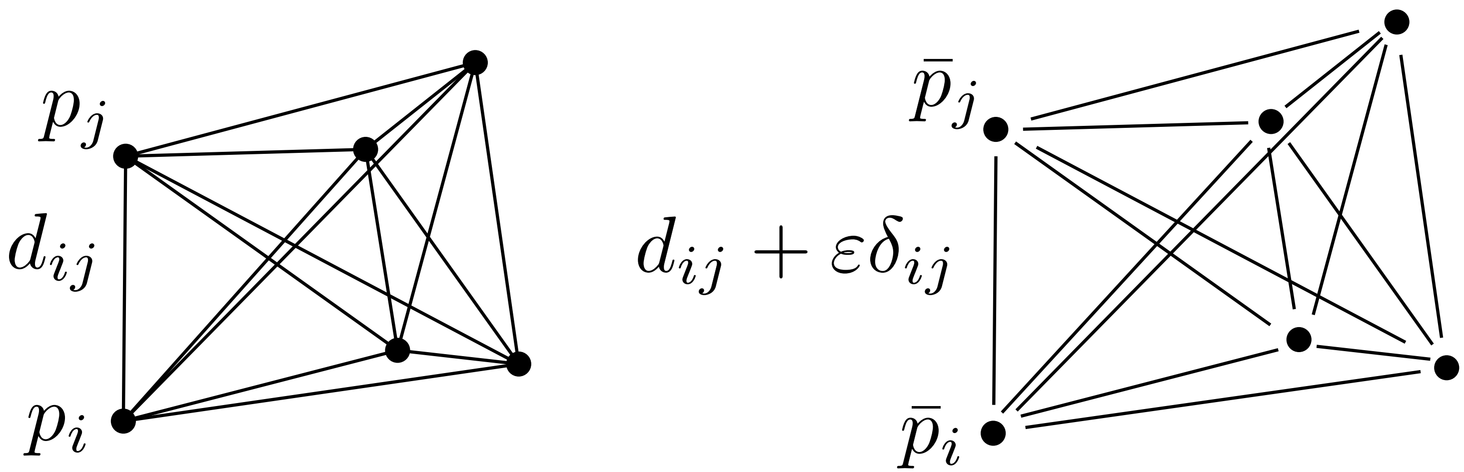
Increasing the distances

$$d_{ij} := \|p_i - p_j\|$$

Find new locations $\bar{p}_i$ such that

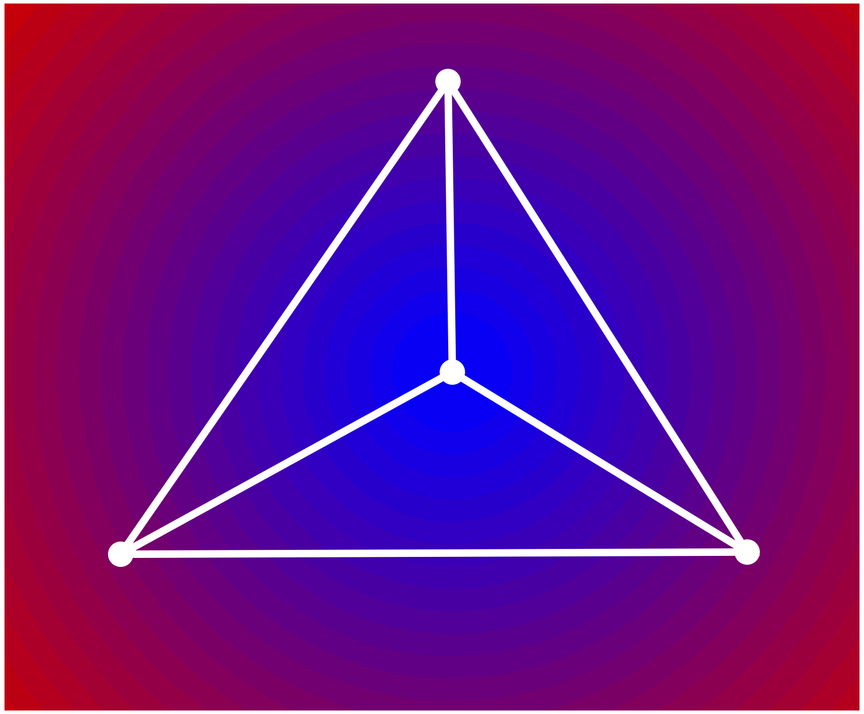
$$\|\bar{p}_i - \bar{p}_j\| \geq d_{ij} + \varepsilon\delta_{ij}$$

for very small (infinitesimal) ε and appropriate numbers δ_{ij}.



If the new distances $d_{ij} + \varepsilon\delta_{ij}$ are generic, the maximal sets of tight inequalities will correspond to minimally rigid graphs.

Heating up the bars



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$$\text{Length increase } \geq \int_{x \in p_i p_j} |x|^2 \, ds$$

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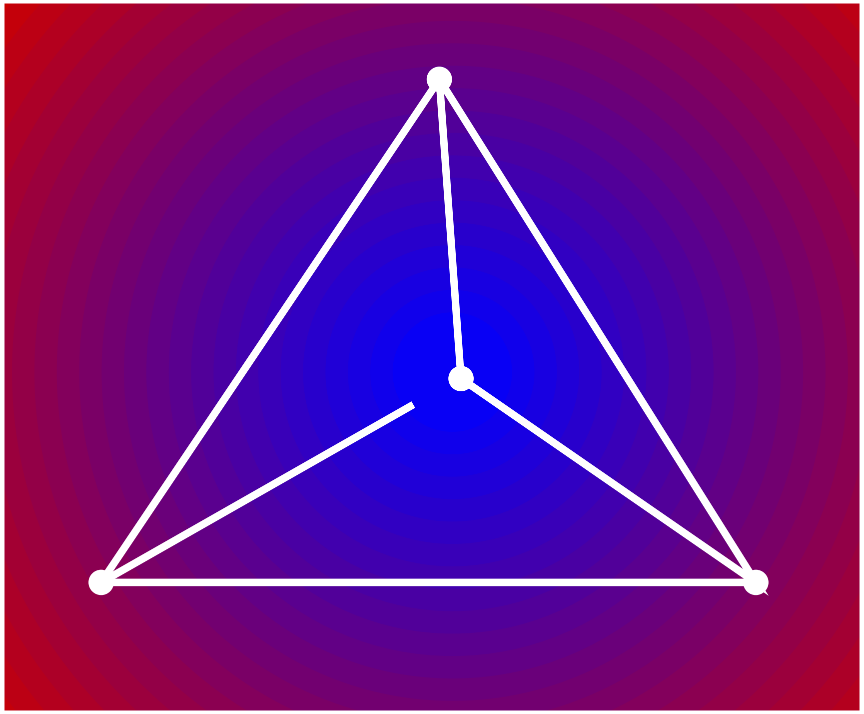

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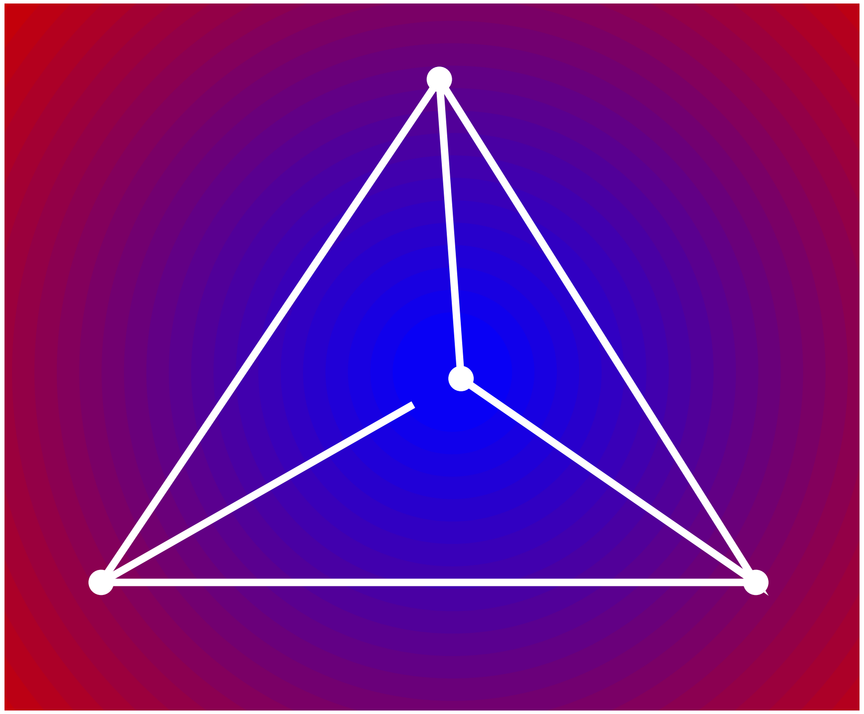

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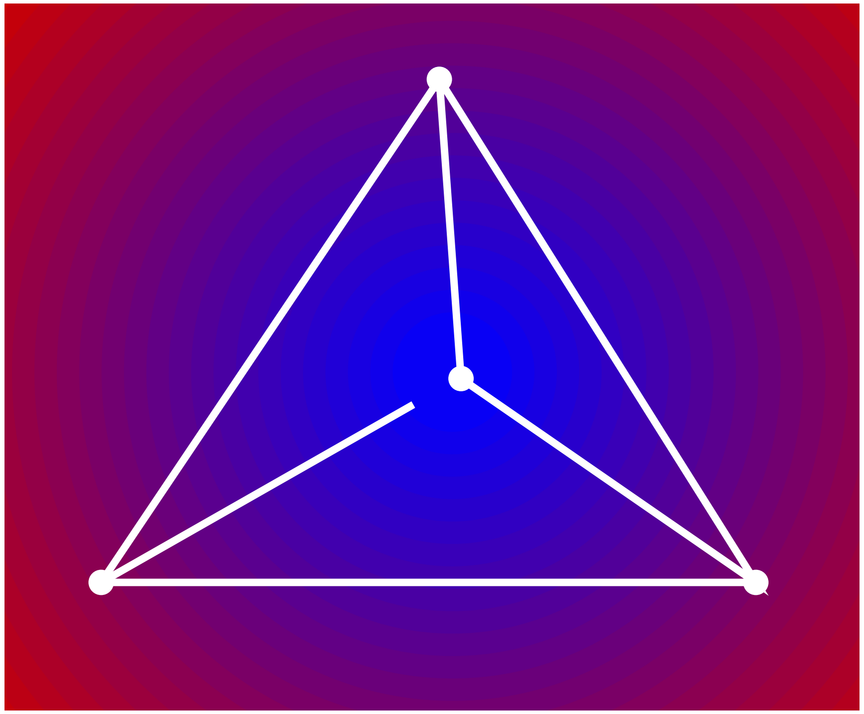

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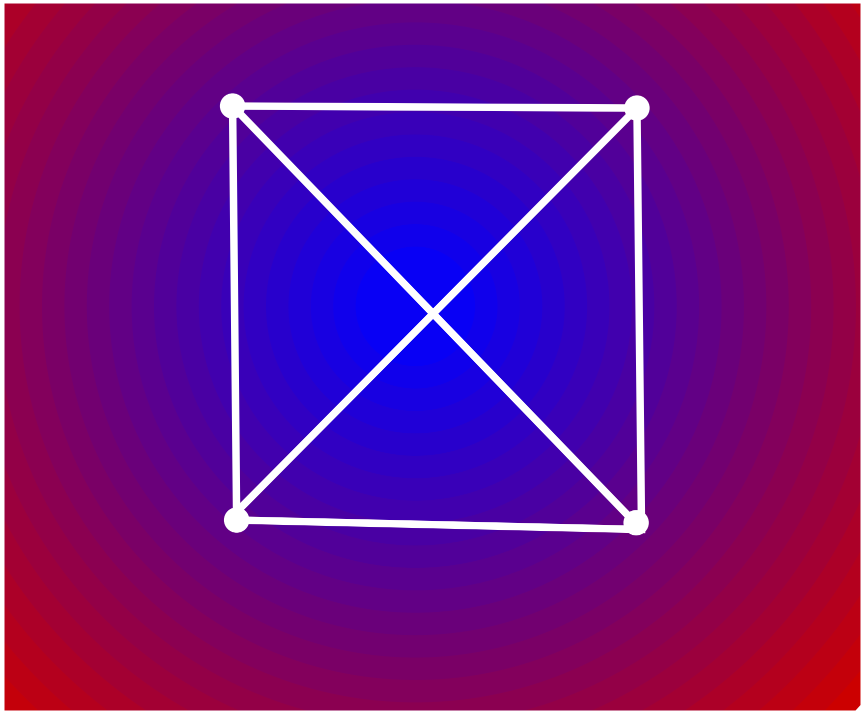
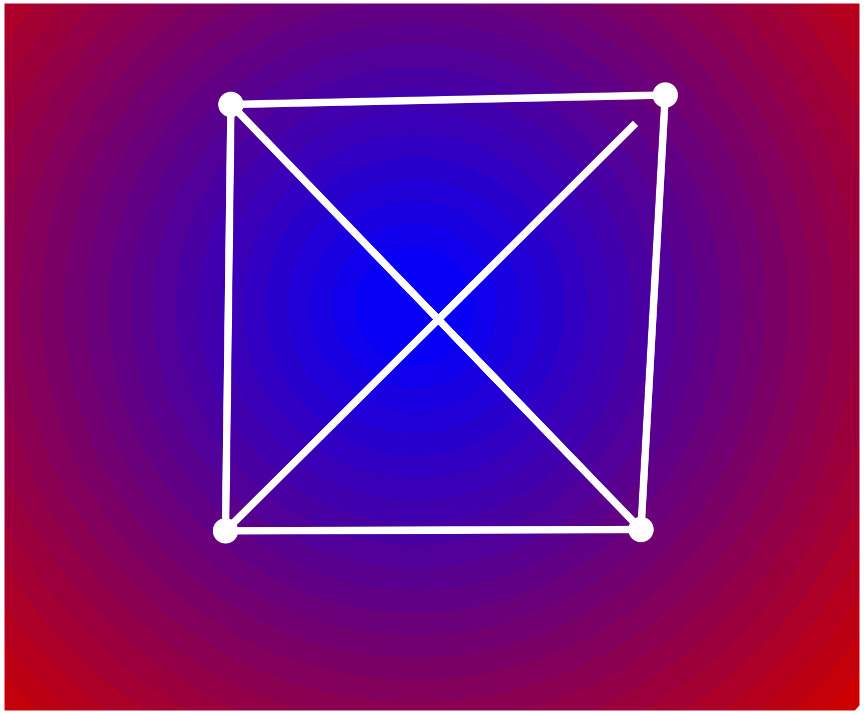
$$\delta_{ij} = |p_i - p_j| \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2) \cdot \tfrac{1}{3}$$

Heating up the bars — points in convex position



$\Rightarrow$



The space of infinitesimal motions

n vertices $p_1, \ldots, p_n$.

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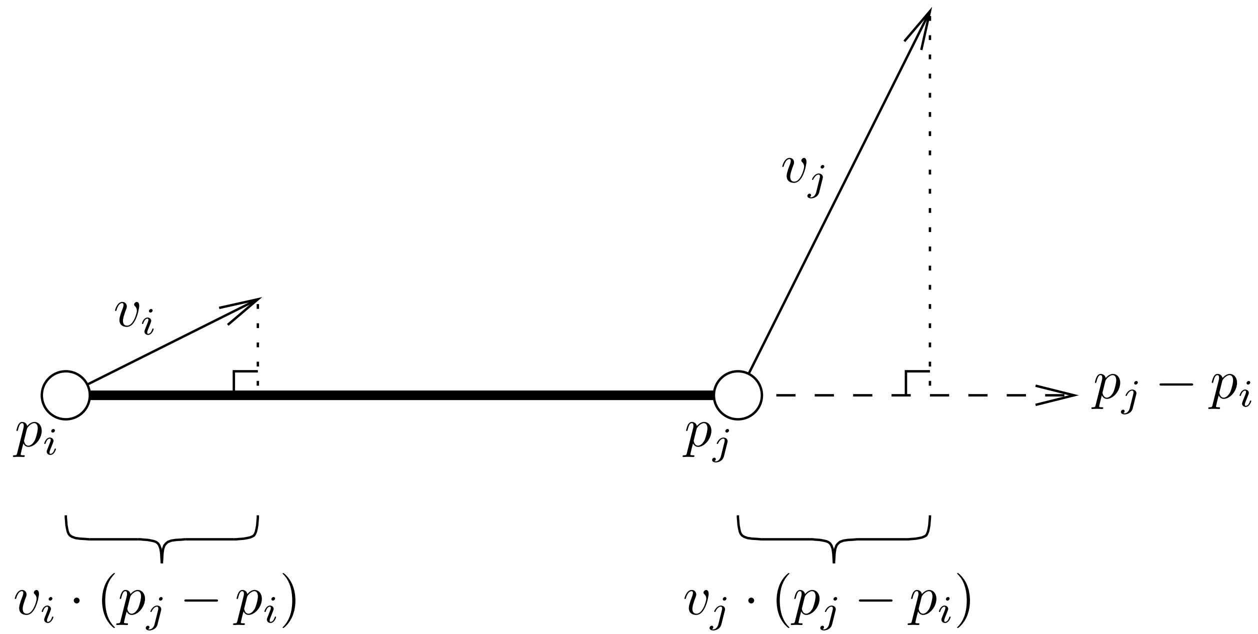
  Velocity vectors $v_1, \ldots, v_n$.

- $\bar{p}_i = p_i + \varepsilon v_i = p_i + dt \cdot v_i$

Expansion

$$\tfrac{1}{2} \cdot \frac{d}{dt} |p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle =: \exp_{ij}$$



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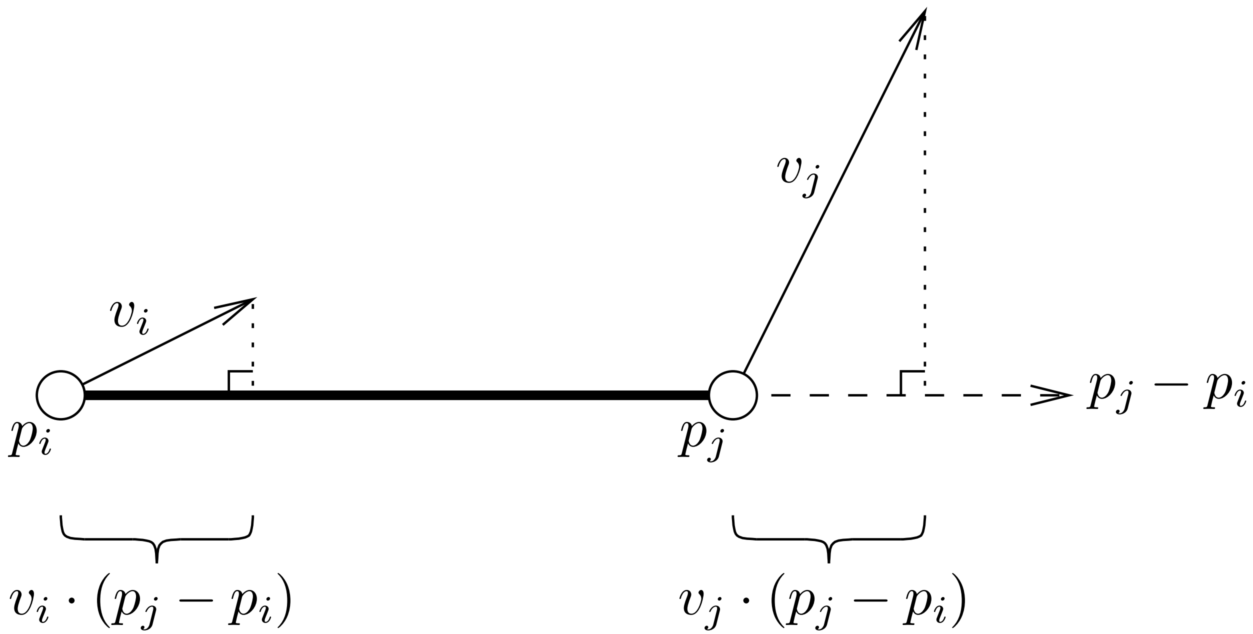


expansion (or *strain*) $\exp_{ij}$ of the segment ij

$$\exp_{ij} = |p_i - p_j| \cdot (\|\bar{p}_i - \bar{p}_j\| - \|p_i - p_j\|)$$

Pinning of Vertices

Trivial Motions: Motions of the point set as a whole (translations, rotations).

Normalization: Pin a vertex and a direction. ("tie-down")

$$v_1 = 0$$

$$v_2 \parallel p_2 - p_1$$

This eliminates 3 degrees of freedom.

The polyhedron lives in $2n - 3$ dimensions.

The PPT polyhedron

$$\bar{X}_f = \{ (v_1, \ldots, v_n) \mid \exp_{ij} \geq f_{ij} \}$$

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

  $[x, y, z] =$ signed area of the triangle xyz
  a, b: two arbitrary points.

Tight edges

For $v = (v_1, \ldots, v_n) \in \bar{X}_f$,

$$E(v) := \{\, ij \mid \exp_{ij} = f_{ij} \,\}$$

is the *set of tight edges* at v.

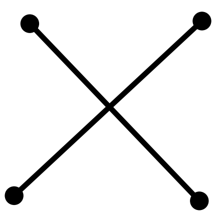
Maximal sets of tight edges $\equiv$ vertices of $\bar{X}_f$.

What are good values of f_{ij}?

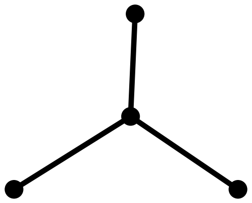
Which configurations of edges can occur in a set of tight edges?

We want:

- no crossing edges



- no 3-star with all angles $\leq 180^\circ$



It is sufficient to look at 4-point subsets.

The PPT-polyhedron

$\longrightarrow$ For every vertex v, $E(v)$ is non-crossing and pointed.

$\longrightarrow |E(v)| \leq 2n - 3$

$\longrightarrow |E(v)| = 2n - 3$ and $\bar{X}_f$ is a simple polyhedron.

Every vertex is incident to $2n - 3$ edges.

Edge $\equiv$ removing a segment from $E(v)$.

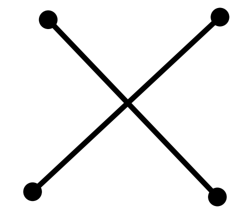
Removing an interior segment leads to an adjacent pseudotriangulation (flip).

Removing a hull segment is an extreme ray. $\square$

Good values f_{ij} for 4 points

In a set of tight edges, we want:

- no crossing edges



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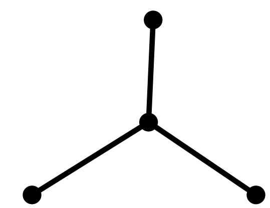

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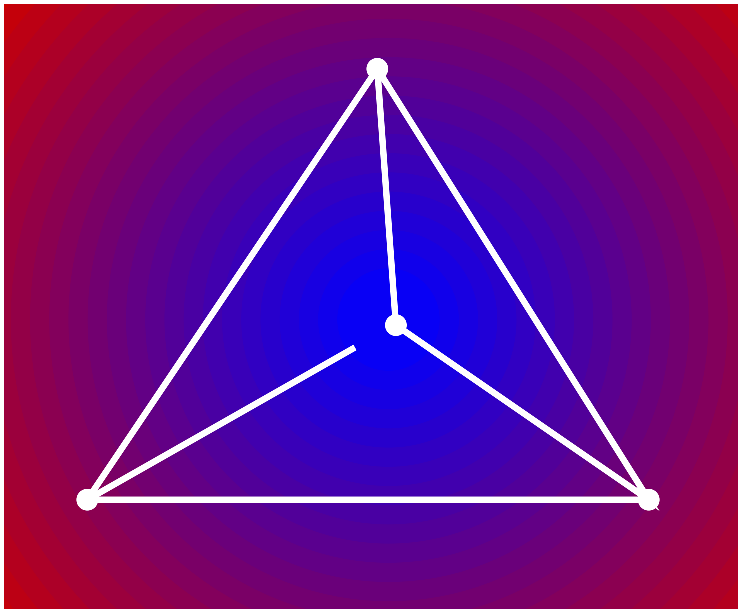


f_{ij} is given on six edges.

Any five values $\exp_{ij}$ determine the last one.

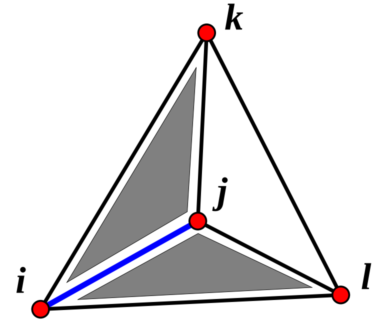
Check if the resulting value $\exp_{ij}$ of the last edge is feasible ($\exp_{ij} \geq f_{ij}$)

$\rightarrow$ checking the sign of an expression.

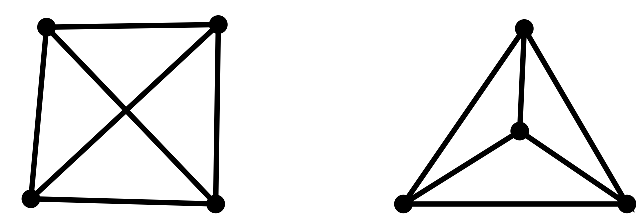
Good Values f_{ij} for 4 points

A 4-tuple p_1, p_2, p_3, p_4 has a unique self-stress (up to a scalar factor).

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}, \text{ for all } 1 \leq i < j \leq 4$$



$\omega_{ij} > 0$ for boundary edges.
$\omega_{ij} < 0$ for interior edges.



Why the stress?

If the *equation*

$$\sum_{1\le i<j\le 4} \omega_{ij} f_{ij} = 0$$

holds, then f_{ij} are the expansion values $\exp_{ij}$ of a motion (v_1, v_2, v_3, v_4).

Actually, "if and only if".

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Actually, "if and only if".

[$M^{\mathrm{T}}\omega = 0$, $f = \exp = Mv$]

Good perturbations

We need

$$\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} > 0$$

for all 4-tuples of points p_1, p_2, p_3, p_4, with

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}, \qquad f_{ij} = [a, p_i, p_j][b, p_i, p_j]$$

Good perturbations

We need

$$\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} > 0$$

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$$\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} = 1$$

Good perturbations

We need

$$\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} > 0$$

for all 4-tuples of points p_1, p_2, p_3, p_4, with

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}, \qquad f_{ij} = [a, p_i, p_j][b, p_i, p_j]$$

$$\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} = 1 > 0$$

What is the meaning of $\sum_{1\leq i<j\leq 4}\omega_{ij}f_{ij}=1$?

"I believe there is some underlying homology in this situation. Given the fact that motions and stresses also fit into a setting of cohomology and homology as well, the authors might, at least, mention possible homology descriptions."

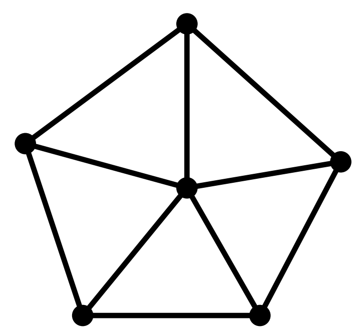
[a referee, about the definition of ω_{ij}]

What is the meaning of $\sum_{1\leq i<j\leq 4} \omega_{ij} f_{ij} = 1$?

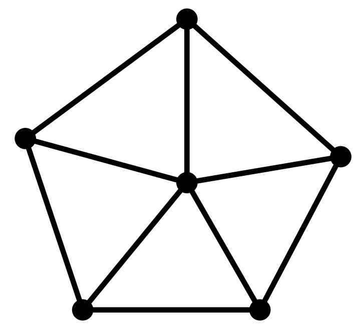
"I believe there is some underlying homology in this situation. Given the fact that motions and stresses also fit into a setting of cohomology and homology as well, the authors might, at least, mention possible homology descriptions."

[a referee, about the definition of ω_{ij}]

One can define a similar formula for ω for the k-wheel.



$\sum_{ij \in E} \omega_{ij} f_{ij} = 1$ for the k-wheel

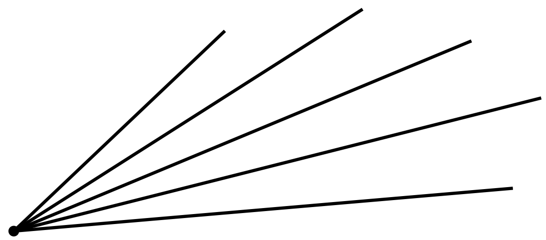


$$\omega_{i,i+1} = \frac{1}{[p_i, p_{i+1}, p_0] \cdot [p_1, p_2, \ldots, p_k]}$$

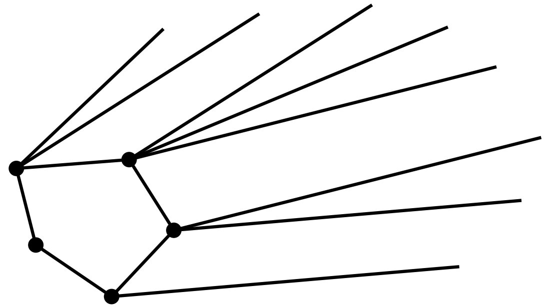
$$\omega_{0i} = \frac{1}{[p_{i-1}, p_i, p_0] \cdot [p_i, p_{i+1}, p_0]} \cdot \frac{[p_{i-1}, p_i, p_{i+1}]}{[p_1, p_2, \ldots, p_k]}$$

Cones and polytopes

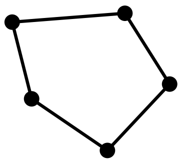
- The *expansion cone*
  $\bar{X}_0 = \{\, \exp_{ij} \geq 0 \,\}$



- The *perturbed expansion cone*
  $=$ the *PPT polyhedron*
  $\bar{X}_f = \{\, \exp_{ij} \geq f_{ij} \,\}$



- The *PPT polytope*
  $X_f = \{\, \exp_{ij} \geq f_{ij},$
  $\exp_{ij} = f_{ij}$ for ij on boundary $\}$



The PPT polytope

Cut out all rays:
Change $\exp_{ij} \geq f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

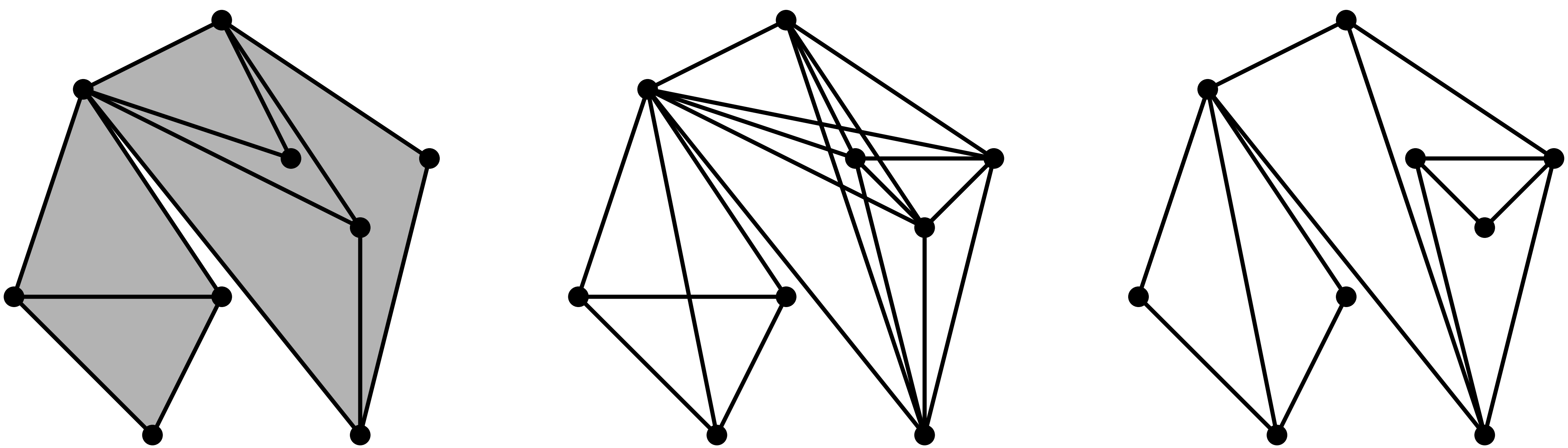
Theorem. *For every set S of points in general position, there is a convex $(2n-3)$-dimensional polytope whose vertices correspond to the pointed pseudotriangulations of S.*

Extreme rays of the expansion cone

The Expansion Cone $\bar{X}_0$:

collapse parallel rays into one ray. $\longrightarrow$ pseudotriangulations minus one hull edge. Rigid subcomponents are identified.

Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]



Expansive motions for a chain (or a polygon)

- Add edges to form a pseudotriangulation
- Remove a convex hull edge
- $\longrightarrow$ expansive mechanism $\square$

Theorem. *Every polygonal arc in the plane can be brought into straight position, without self-overlap.*

Every polygon in the plane can be unfolded into convex position.

[Connelly, Demaine, Rote 2001], [Streinu 2001]

The PT polytope

Vertices correspond to *all* pseudotriangulations, pointed or not.

Change inequalities $\exp_{ij} \geq f_{ij}$ to

$$\exp_{ij} + (s_i + s_j)\|p_j - p_i\| \geq f_{ij}$$

with a "slack variable" s_i for every vertex.
$s_i = 0$ indicates that vertex i is pointed.

A "flip" may insert an edge, changing a vertex from pointed to non-pointed, or vice versa.

Faces are in one-to-one correspondence with all non-crossing graphs.

[Orden, Santos 2002]

Which f_{ij} to choose?

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

Go to the space of the (exp_{ij}) variables instead of the (v_i) variables.

$$\mathrm{exp} = Mv$$

Characterization of the space $(\exp_{ij})_{i,j}$

A set of values $(\exp_{ij})_{1 \le i < j \le n}$ forms the expansion values of a motion $(v_1, \ldots, v_n)$ if and only if the equation

$$\sum_{1 \le i < j \le 4} \omega_{ij} \exp_{ij} = 0$$

holds for all 4-tuples.

SKIP

A canonical representation

$$\sum_{1 \le i < j \le 4} \omega_{ij} \exp_{ij} = 0, \text{ for all 4-tuples}$$

$$\exp_{ij} \ge f_{ij}, \text{ for all pairs } i, j$$

A canonical representation

$$\sum_{1\leq i<j\leq 4} \omega_{ij} \exp_{ij} = 0, \text{ for all 4-tuples}$$

$$\exp_{ij} \geq f_{ij}, \text{ for all pairs } i, j$$

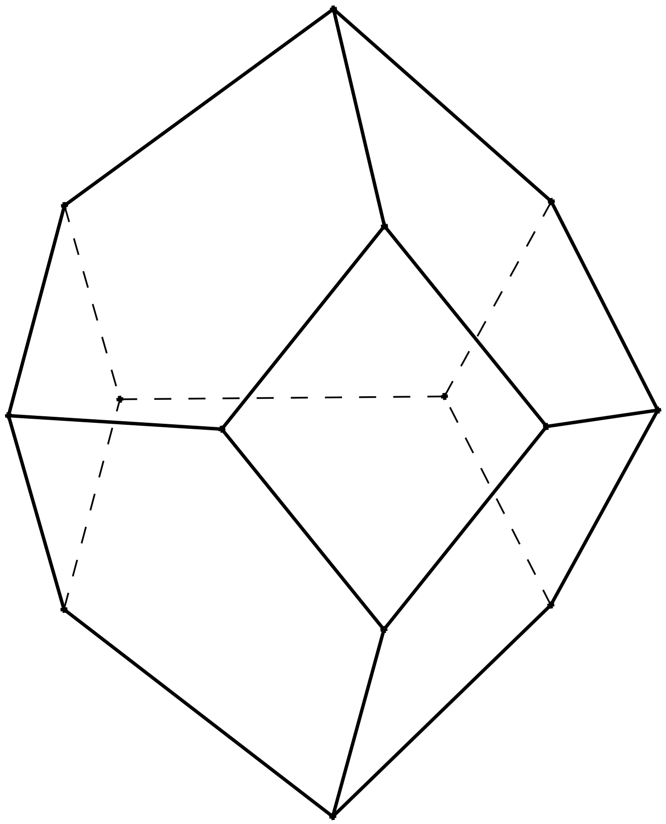
$$\sum_{1\leq i<j\leq 4} \omega_{ij} f_{ij} = 1, \text{ for all 4-tuples}$$

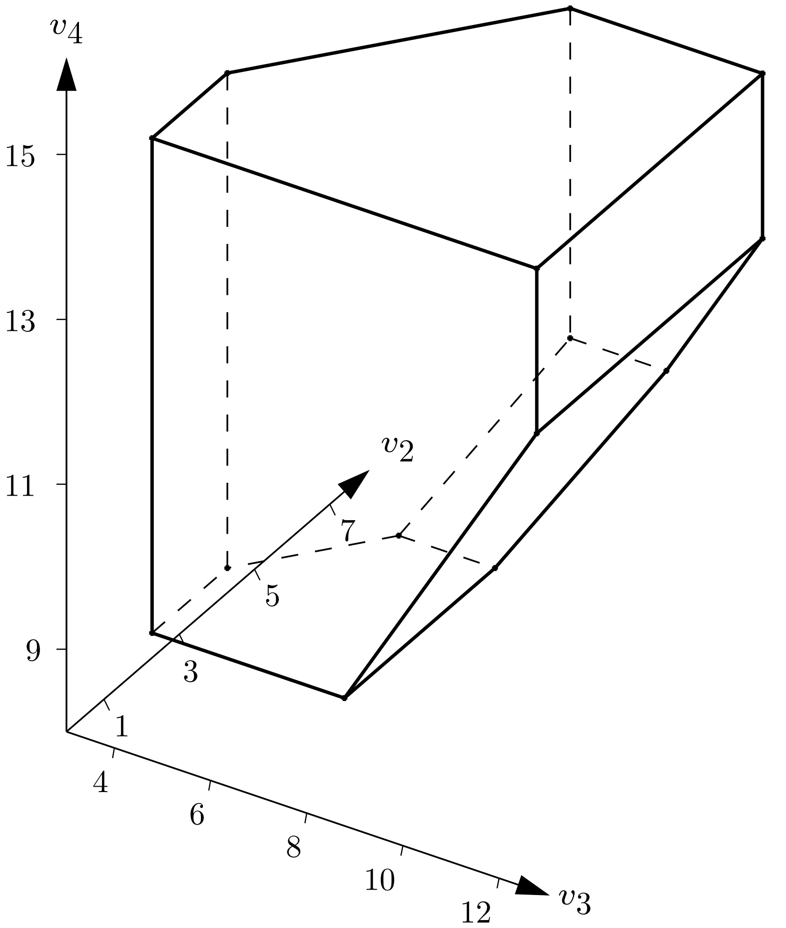
Substitute $d_{ij} := \exp_{ij} - f_{ij}$:

$$\textstyle\sum_{1\leq i<j\leq 4} \omega_{ij} d_{ij} = -1, \text{ for all 4-tuples} \quad (1)$$

$$d_{ij} \geq 0, \text{ for all } i, j \quad (2)$$

The associahedron





Catalan structures

- Triangulations of a convex polygon / edge flip
- Binary trees / rotation
- $(a * (b * (c * d))) * e$ / $((a * b) * (c * d)) * e$
- . . . .

The secondary polytope

Triangulation $T \mapsto (a_1, \ldots, a_n)$.

$$a_i := \text{total area of all triangles incident to } p_i$$

vertices $\equiv$ regular triangulations of $(p_1, \ldots, p_n)$

$(p_1, \ldots, p_n)$ in convex position:
pseudotriangulations $\equiv$ triangulations $\equiv$ regular triangulations.

$\longrightarrow$ two realizations of the associahedron.

These two associahedra are affinely equivalent.

Expansive motions in one dimension

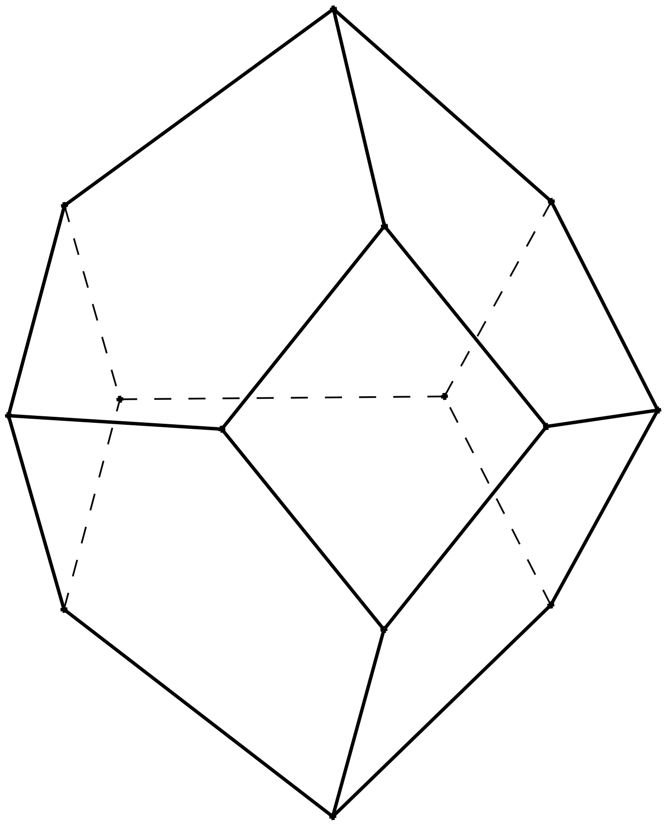
$$\{\,(v_i) \in \mathbb{R}^n \mid v_j - v_i \geq f_{ij} \text{ for } 1 \leq i < j \leq n\,\}$$

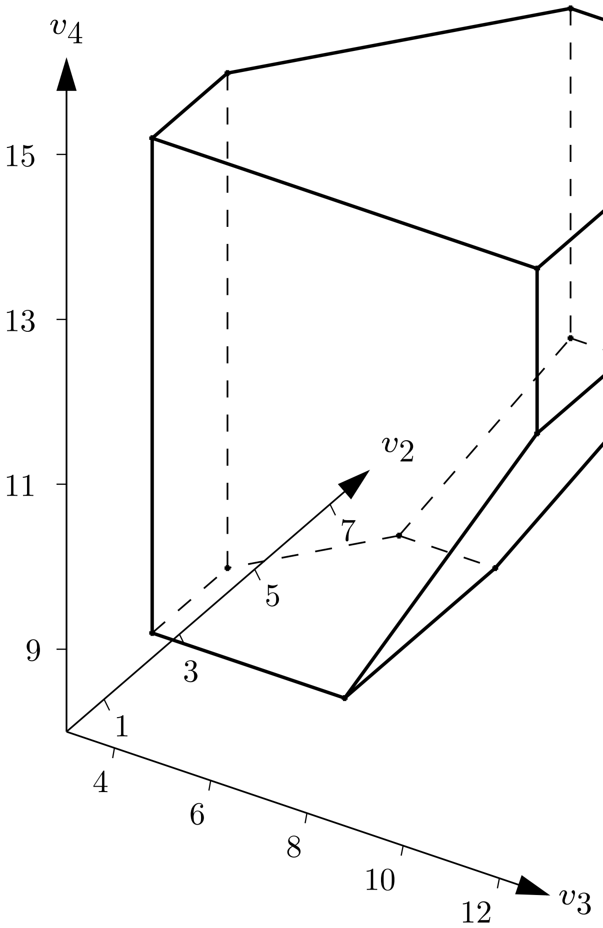
For example, $f_{ij} := (i-j)^2$.

$\longrightarrow$ gives rise to *different* realizations of the associahedron.

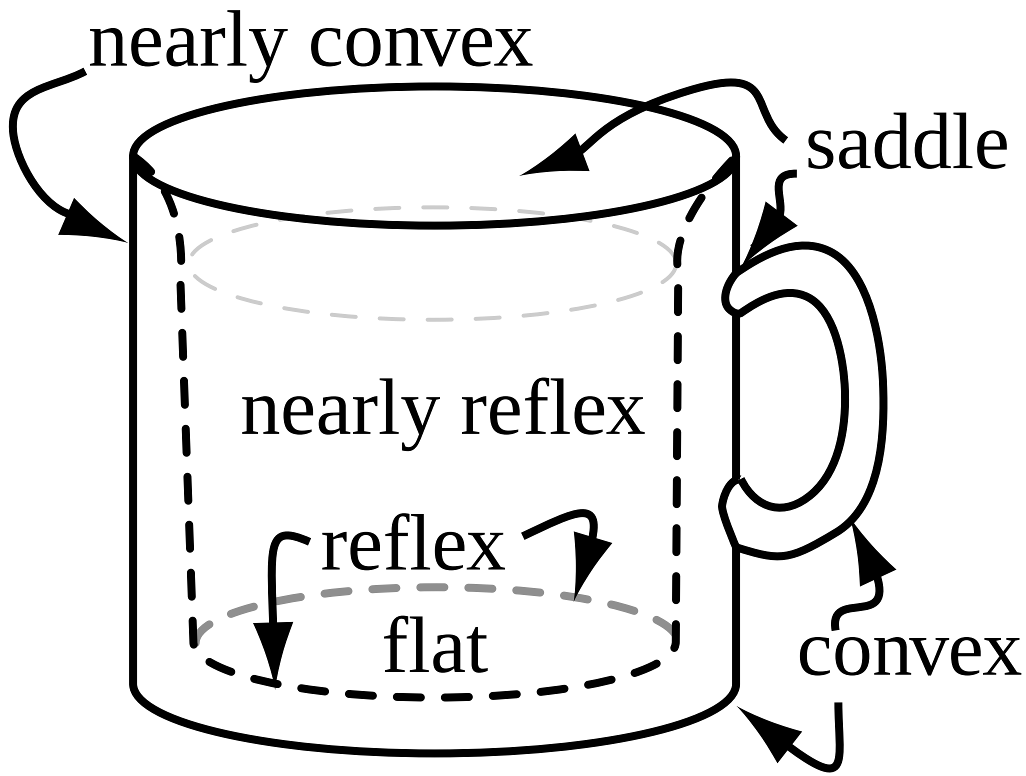
[Gelfand, Graev, and Postnikov 1997], in a dual setting.
[Postnikov 1997], [Zelevinsky ?], [Stasheff 1997]

The associahedron





3. Locally convex surfaces
Motivation: the reflex-free hull

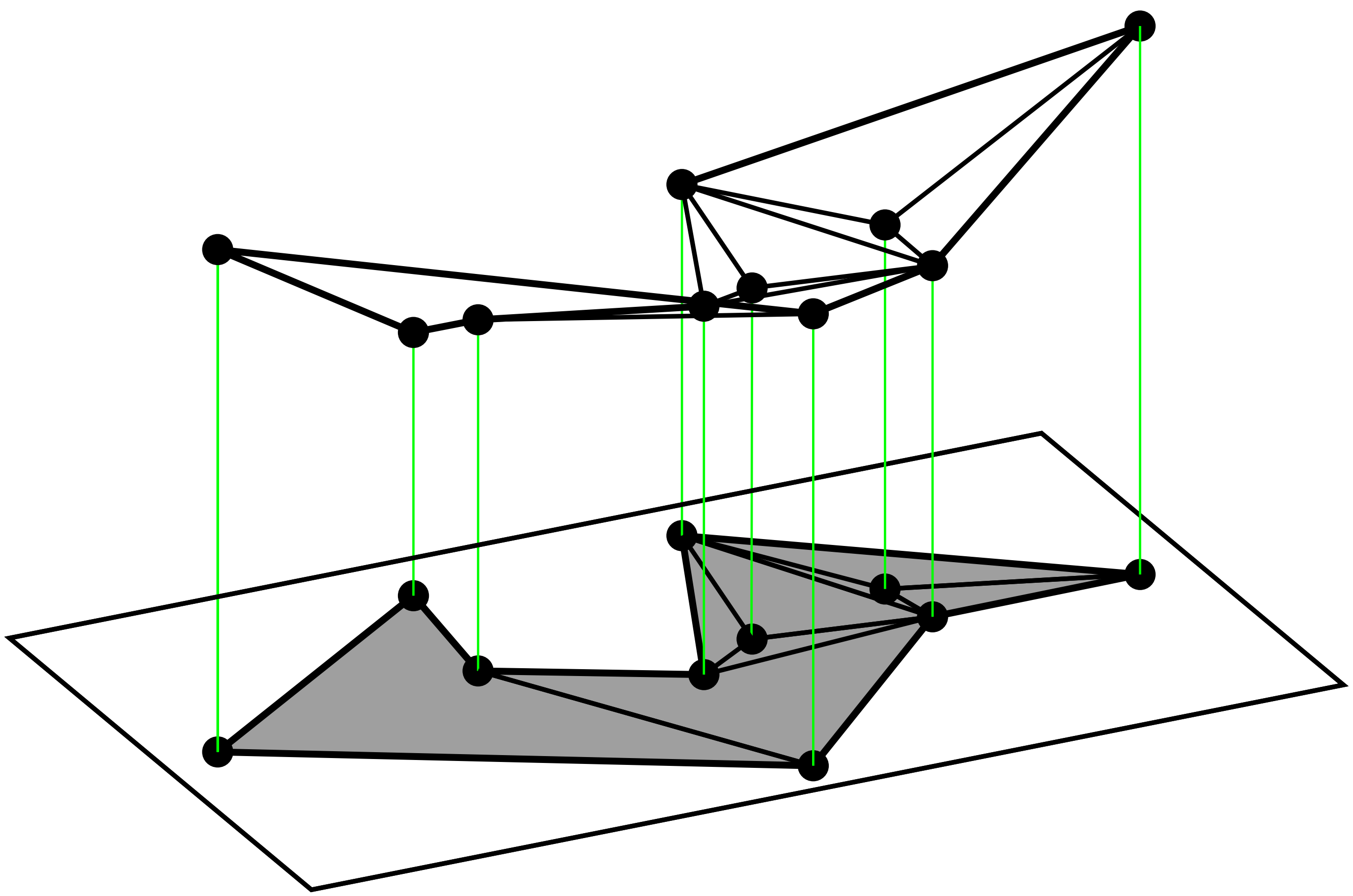


an approach for recognizing pockets in biomolecules

[Ahn, Cheng, Cheong, Snoeyink 2002]

Locally convex functions

A function over a polygonal domain P is *locally convex* if it is convex on every segment in P.



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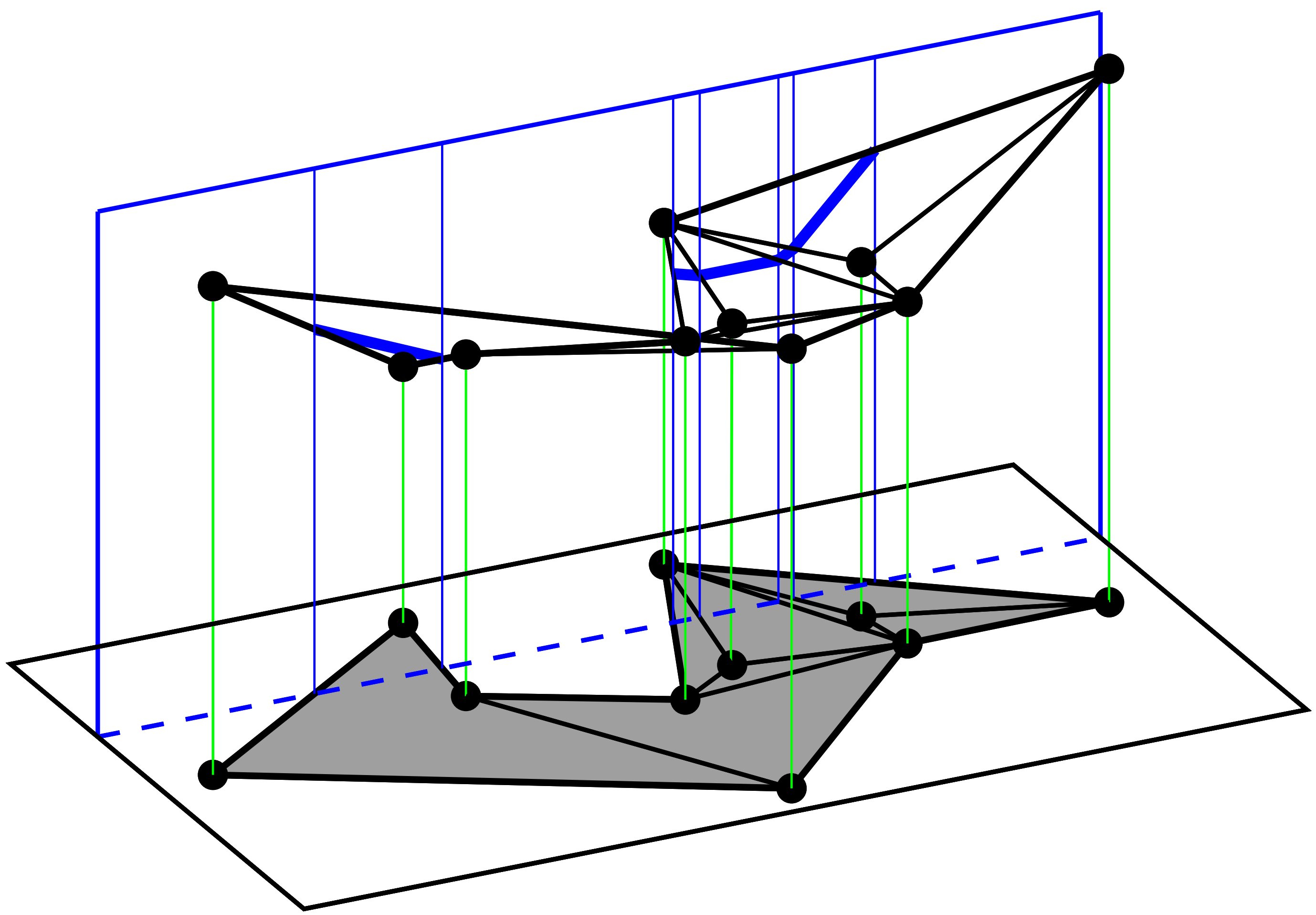


Locally convex functions on a poipogon

A *poipogon* (P, S) is a simple polygon P with some additional vertices inside.

Given a poipogon and a height value h_i for each $p_i \in S$, find the highest locally convex function $f\colon P \to \mathbb{R}$ with $f(p_i) \leq h_i$.

If P is convex, this is the lower convex hull of the three-dimensional point set (p_i, h_i).

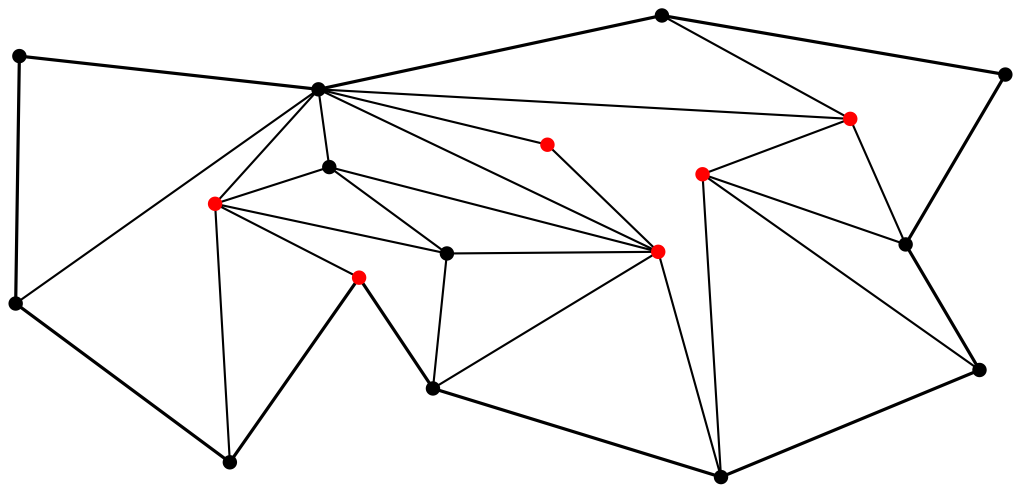
In general, the result is a piecewise linear function defined on a pseudotriangulation of (P, S). (Interior vertices may be missing.)

$\rightarrow$ *regular pseudotriangulations*

[Aichholzer, Aurenhammer, Braß, Krasser 2003]

The surface theorem

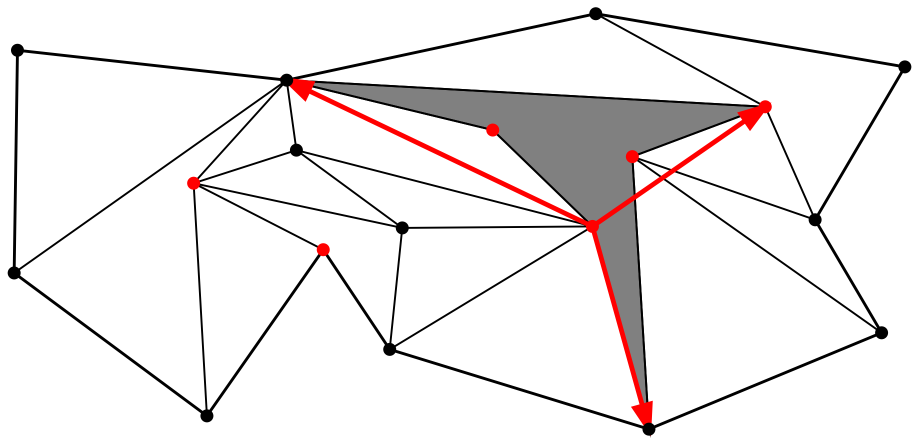
In a pseudotriangulation T of (P, S), a vertex is *complete* if it is a corner in all pseudotriangulations to which it belongs.



Theorem. *For any given set of heights* h_i *for the complete vertices, there is a unique piecewise linear function* f *on the pseudotriangulation with theses heights. The function depends monotonically on the given heights.*

In a triangulation, all vertices are complete.

Proof of the surface theorem



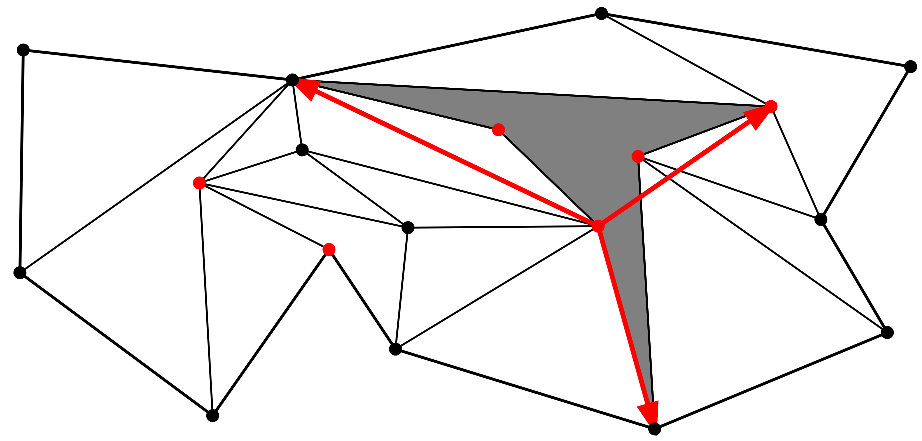
Each incomplete vertex p_i is a convex combination of the three corners of the pseudotriangle in which its large angle lies:

$$p_i = \alpha p_j + \beta p_k + \gamma p_l, \text{ with } \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma > 0.$$

$$\rightarrow h_i = \alpha h_j + \beta h_k + \gamma h_l$$

h is a *harmonic function* on the incomplete vertices.

Proof of the surface theorem



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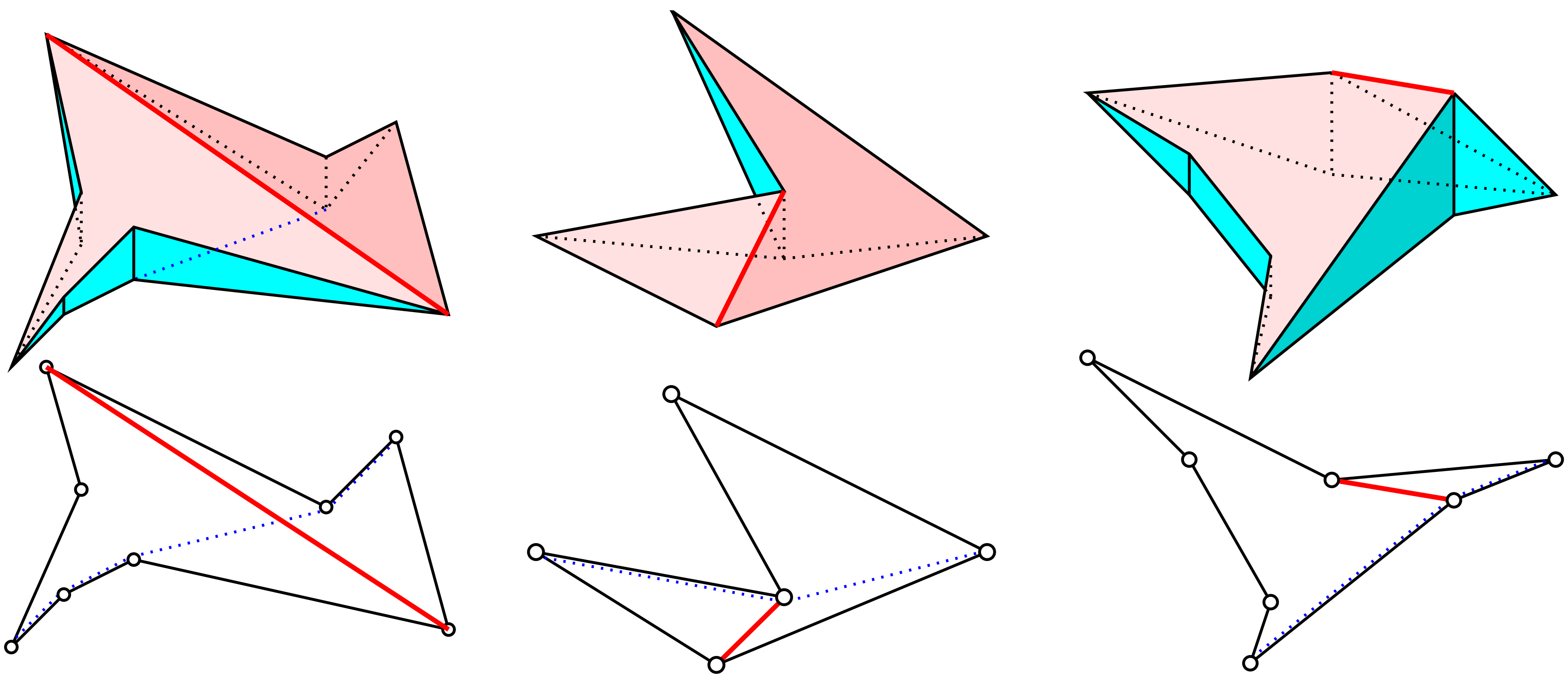
$$p_i = \alpha p_j + \beta p_k + \gamma p_l, \text{ with } \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma > 0.$$
$$\to h_i = \alpha h_j + \beta h_k + \gamma h_l$$

h is a *harmonic function* on the incomplete vertices. The coefficient matrix of the mapping $M\colon (h_1, \ldots, h_n) \mapsto (h'_1, \ldots, h'_n)$ is a stochastic matrix. M is a monotone function, and M^n is a contraction. $\to$ There is always a unique solution.

Flipping to optimality

Find an edge where convexity is violated, and flip it.



convexifying flips

a planarizing flip

A flip has a non-local effect on the whole surface.

The surface moves down monotonically.

Realization as a polytope

Theorem. *There exists a convex polytope whose vertices are in one-to-one correspondence with the regular pseudotriangulations of a poipogon, and whose edges represent flips.*

[Aichholzer, Aurenhammer, Braß, Krasser 2003]

Pseudotriangulation $T \mapsto (a_1, \ldots, a_n)$:

$$\int_P f(x,y)\,dx\,dy = a_1 h_1 + \cdots + a_n h_n$$

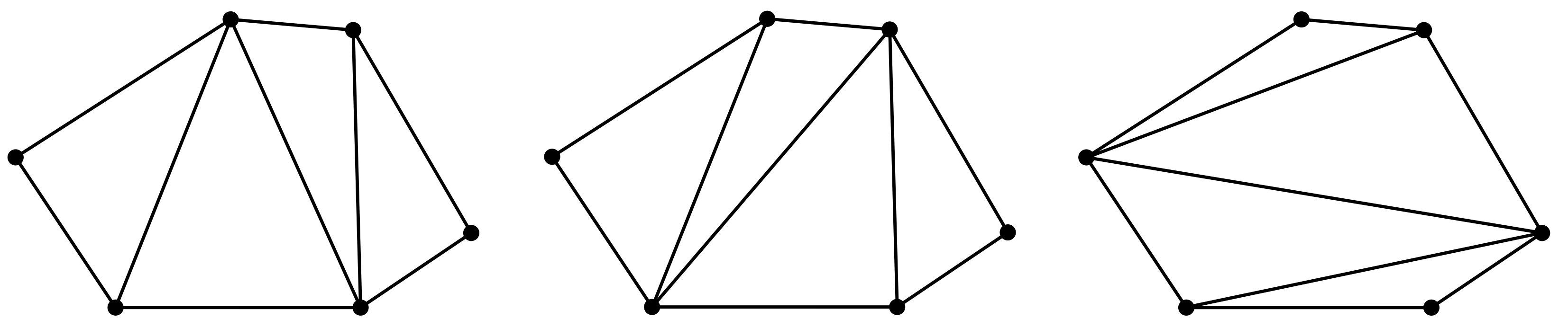
($a_i = 0$ for all incomplete vertices p_i.)
T is represented by the point $(a_1, \ldots, a_n) \in \mathbb{R}^n$.

For a simple polygon (without interior points), all pseudotriangulations are regular.

4. Canonical pseudotriangulations

Maximize/minimize $\sum_{i=1}^{n} c_i \cdot v_i$ over the PPT-polytope.

$c_i := p_i$:



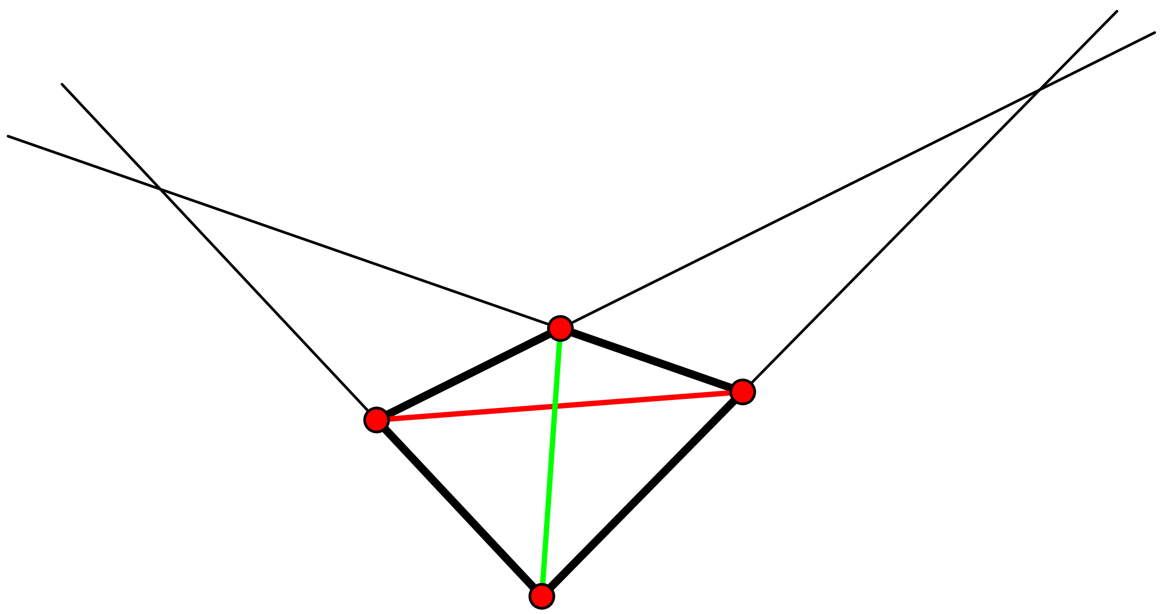
(a) (b) (c)

Delaunay triangulation

Max/Min $\sum p_i \cdot v_i$
(affinely invariant)

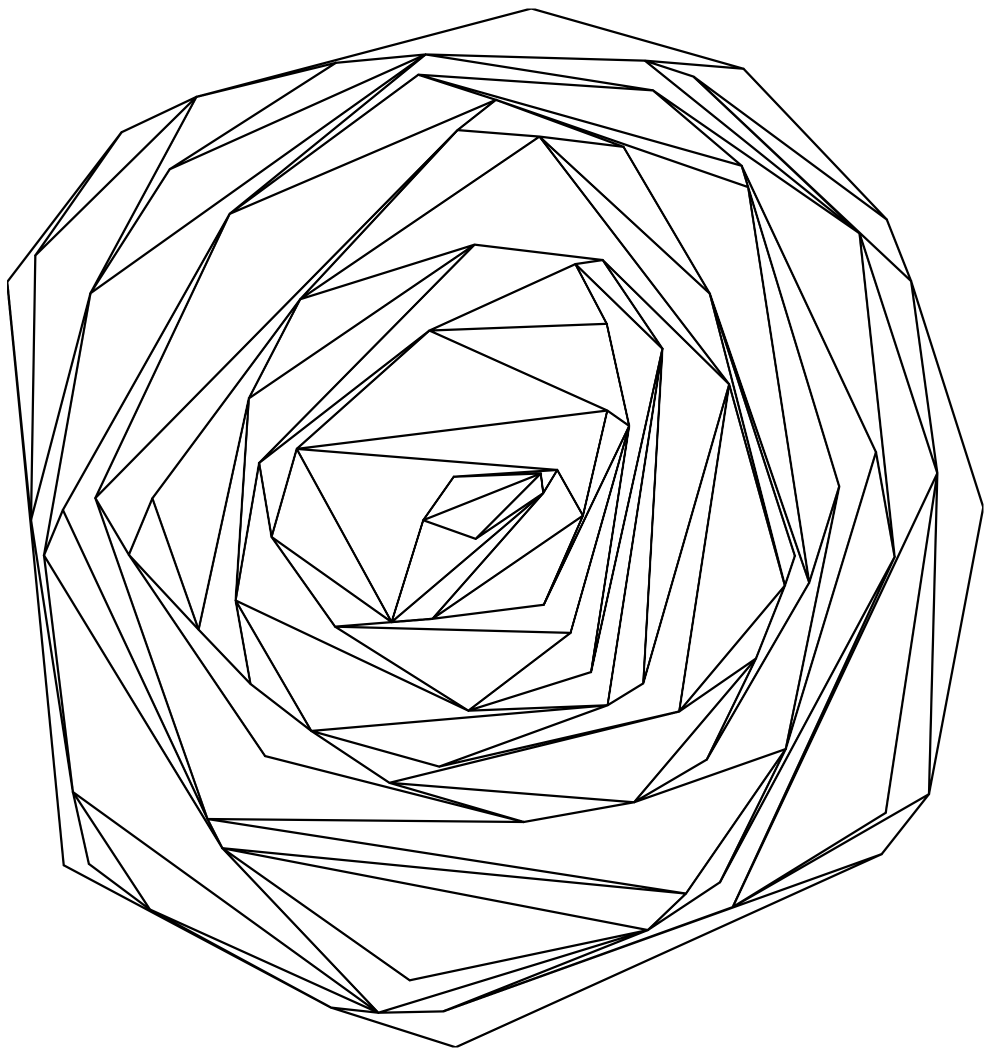
(Can be constructed as the lower/upper convex hull of lifted points.) [André Schulz]

Edge flipping criterion for canonical pseudotriangulations of 4 points in convex position

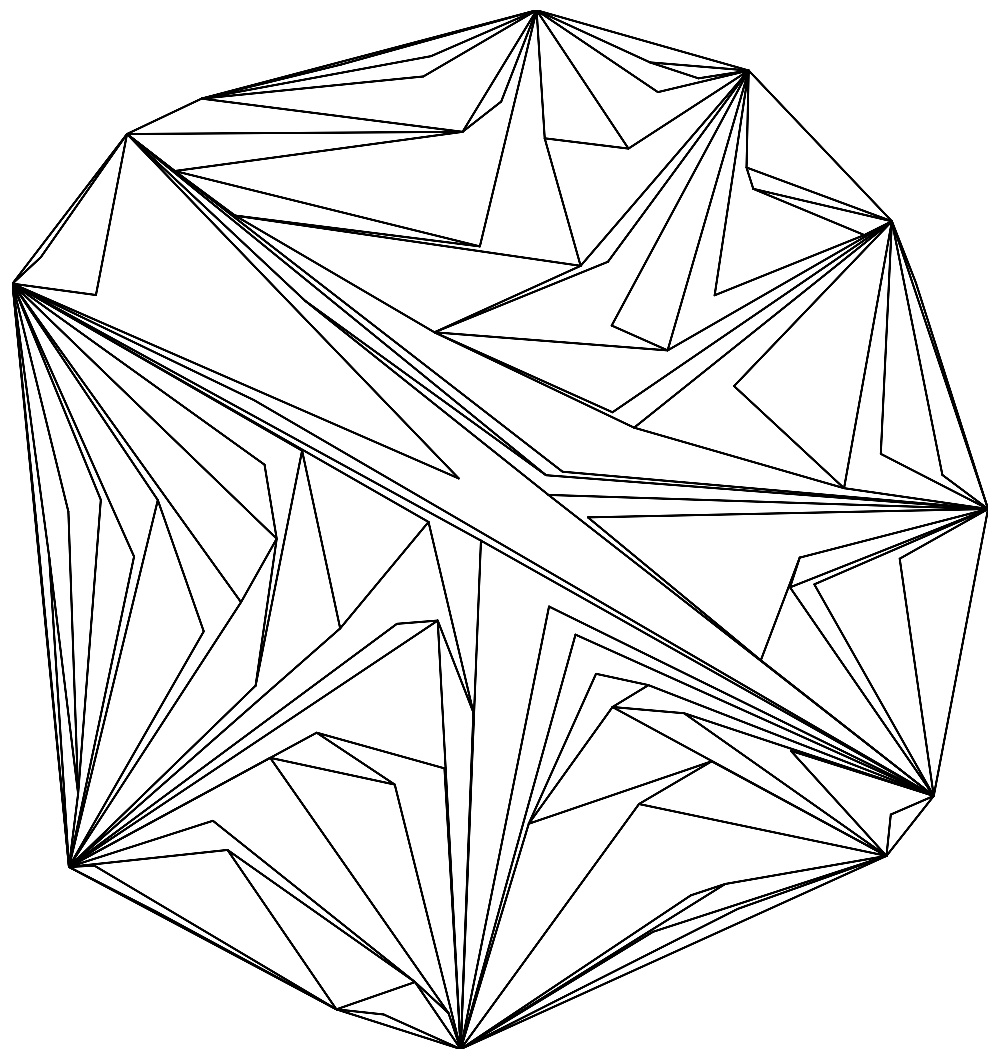


Maximize/minimize the product of the areas. (Also for 4 points in non-convex position)
Invariant under affine transformations.

The “Delone pseudotriangulation” for 100 random points



The “Anti-Delone pseudotriangulation” for 100 random points



The Maxwell-Cremona Correspondence [1864/1872]

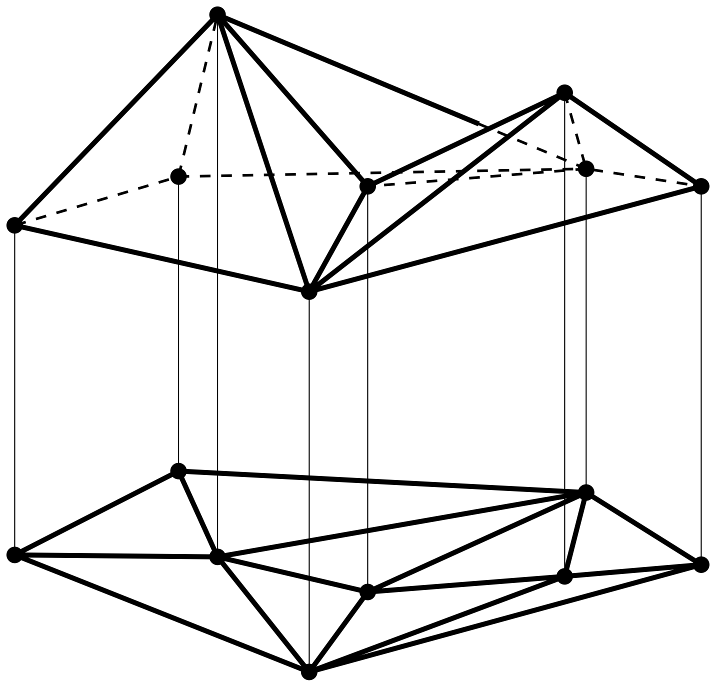
self-stresses on a planar framework

$\Updownarrow$ one-to-one correspondence

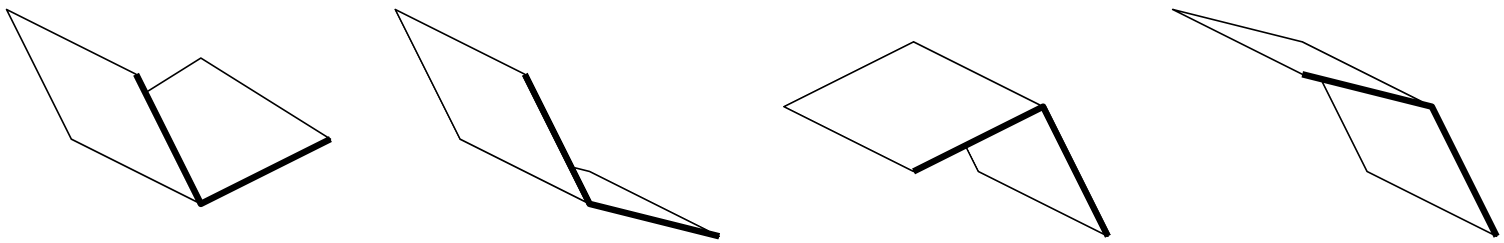
reciprocal diagram

$\Updownarrow$ one-to-one correspondence

3-d lifting (polyhedral terrain)



Valley and mountain folds



$\omega_{ij} > 0$ | $\omega_{ij} < 0$

valley | mountain

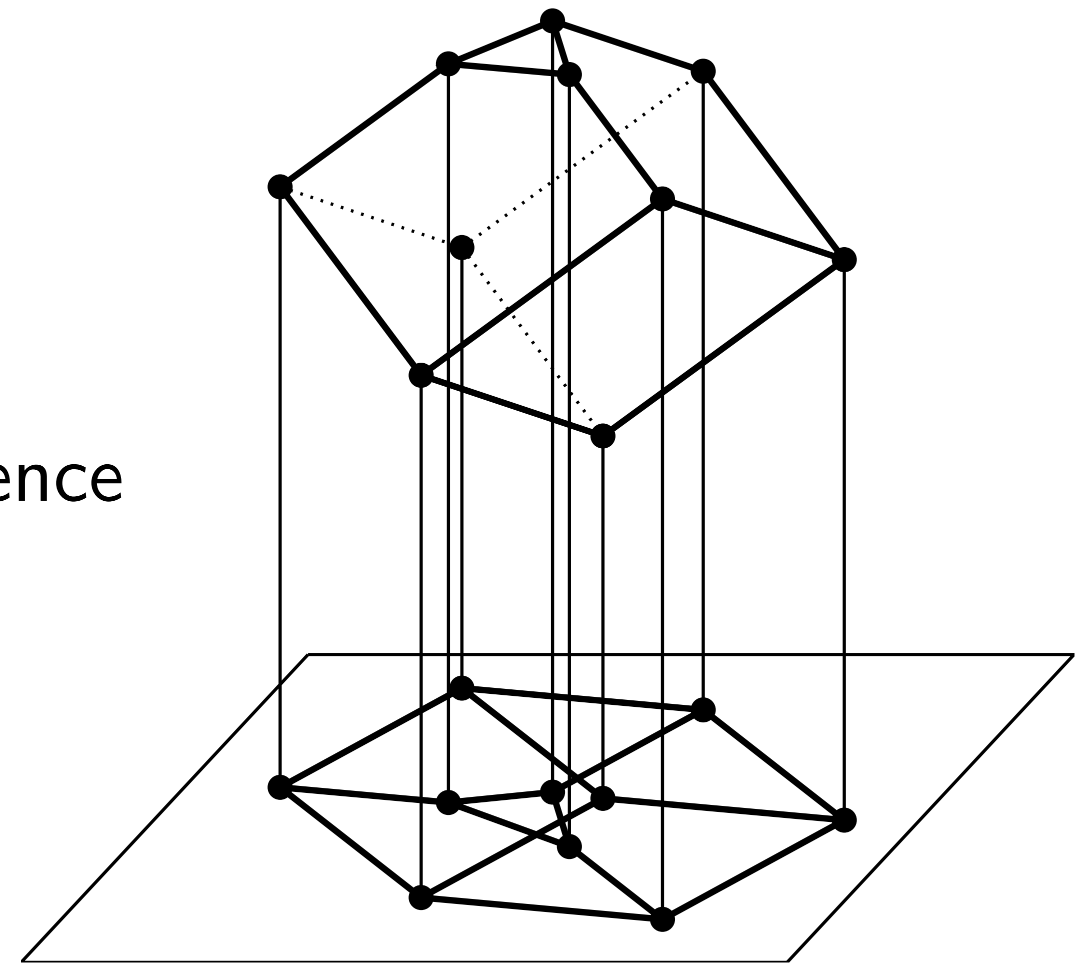
bar or strut | bar

The Maxwell-Cremona Correspondence for closed polyhedral surfaces

3-d lifting
(spherical polyhedral surface)

⇕ one-to-one correspondence

self-stresses on a framework which is a planar graph



Geometric construction of the Delone pseudotriangulation for convex position

[Günter Rote, André Schulz]

$$\begin{array}{ll} \text{minimize} & \langle v_i, p_i \rangle \\ \text{subject to} & \langle v_i - v_j, p_i - p_j \rangle \geq f_{ij} \\ & \sum v_i = 0 \end{array}$$

Consider the dual linear program in variables $\omega_{ij} = \omega_{ji} \geq 0$.

$$\begin{array}{ll} \text{maximize} & \text{some objective function} \\ \text{subject to} & \sum_j \omega_{ij}(p_j - p_i) = \bar{p} - p_i, \text{ for all } i \\ & \omega_{ij} \geq 0. \end{array}$$

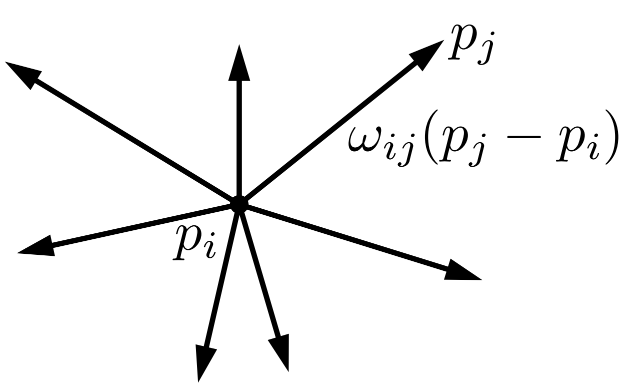
with $\bar{p} = \sum p_i / n =$ center of gravity.

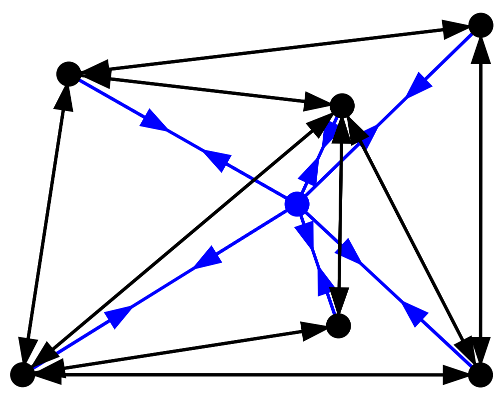
The dual variables are stresses

$$\sum_j \omega_{ij}(p_j - p_i) = \bar{p} - p_i$$

$\omega_{ij} = \omega_{ji} \in \mathbb{R}$ are *stresses* on the edges.
Consider $p_0 := \bar{p}$ as an additional vertex with $\omega_{0i} = -1$:
Equilibrium of forces in vertex i:

$$\sum_{j=0}^{n} \omega_{ij}(p_j - p_i) = 0$$

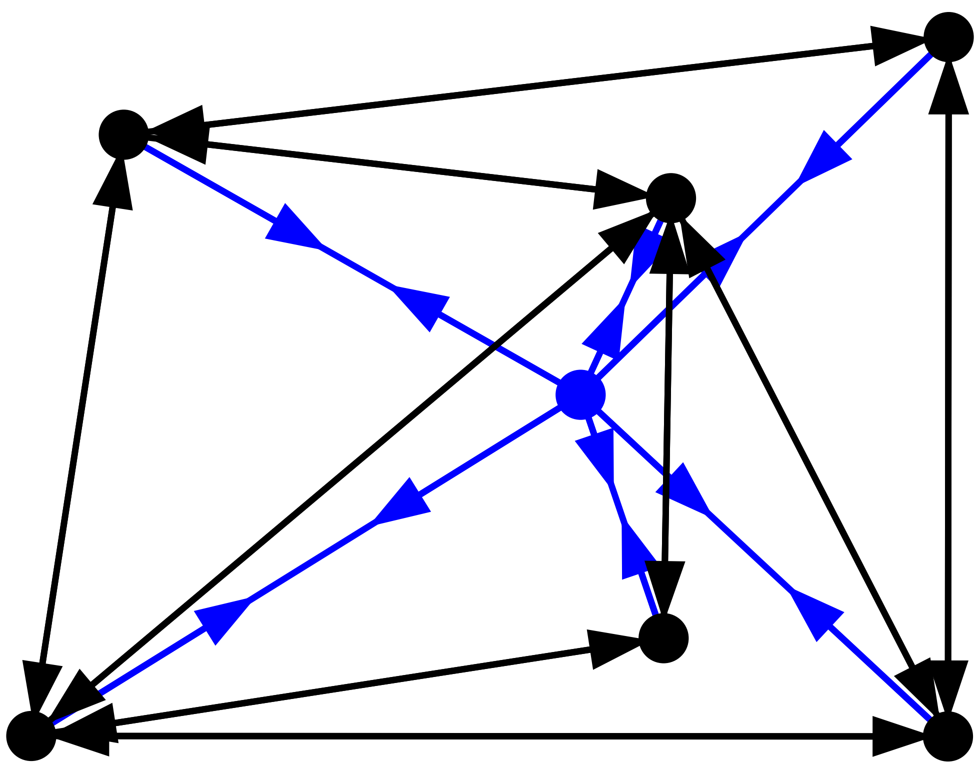




Stresses

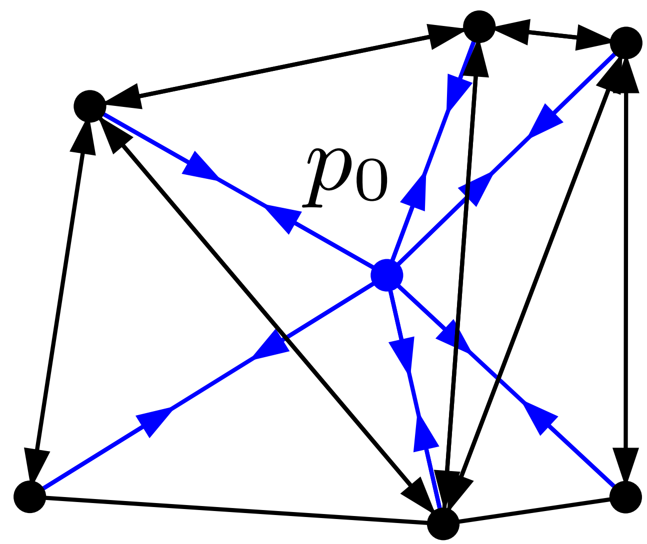
The optimum primal solution will have $\langle v_i - v_j, p_i - p_j \rangle = f_{ij}$ on some pseudotriangulation $E(v)$.

Complementary slackness implies that $\omega_{ij} = 0$ for $ij \notin E(v)$.



Stresses in the convex case

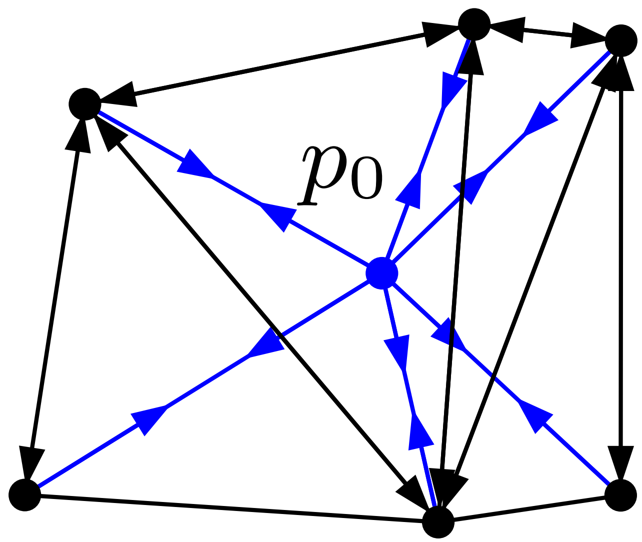
$E(v)$ together with the additional edges $p_i p_0$ is a planar graph.



Maxwell-Cremona theorem $\rightarrow$ lifting of a polytope:
Overlay of

- a convex lifting of the triangulation $E(v)$ and
- a pyramid formed by p_0 and the convex polygon $p_1 p_2 \dots p_n$.

The lifting in the convex case

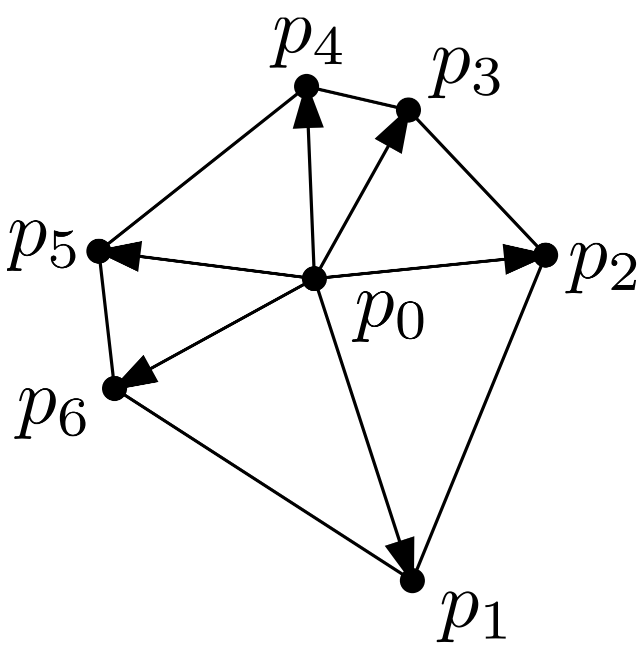


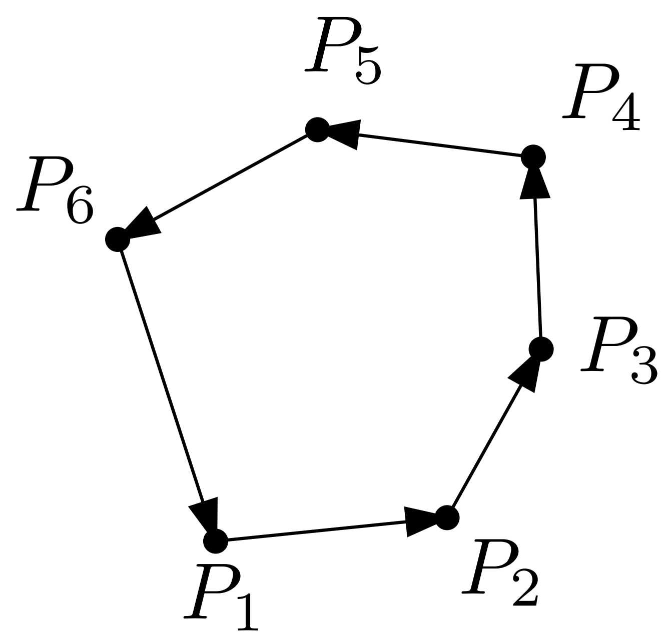
The stresses on the spokes p_0p_i are known ($\omega_{0i} = -1$)
$\rightarrow$ the heights of p_1, p_2, . . . , p_n can be computed.
The lower convex hull of these points gives the “Delone” (pseudo-)triangulation.
The upper convex hull of the same lifted points gives the “anti-Delone” (pseudo-)triangulation.

Calculation of the heights

Let $p_1 p_2 \ldots p_n$ be a convex polygon.
$\sum(p_i - p_0) = 0$ by definition.
Form a new "sum polygon" whose sides are $p_i - p_0$:

$$P_i - P_{i-1} = p_i - p_0$$





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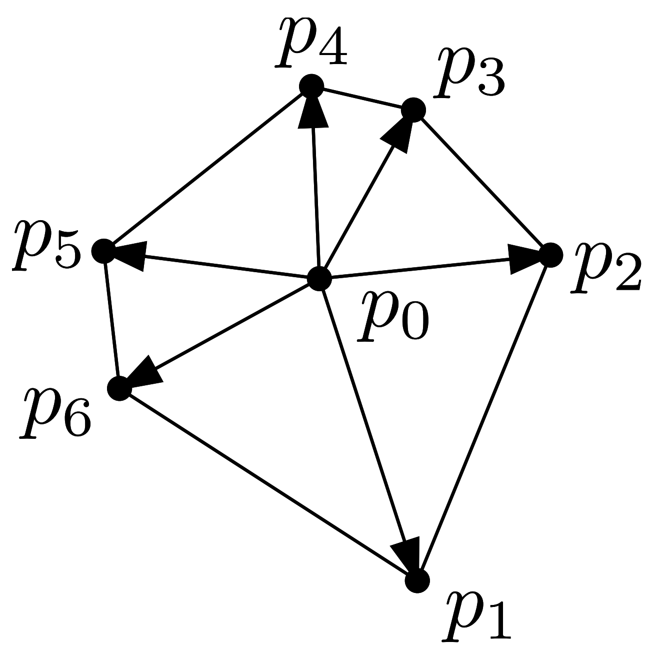
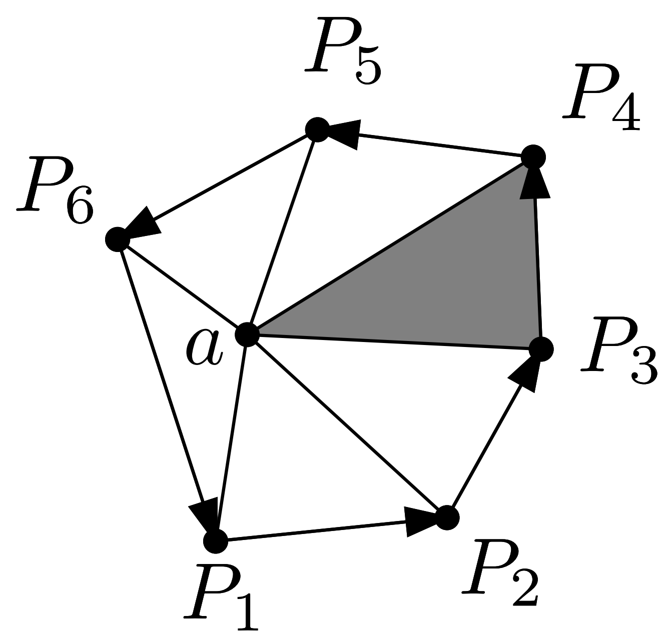
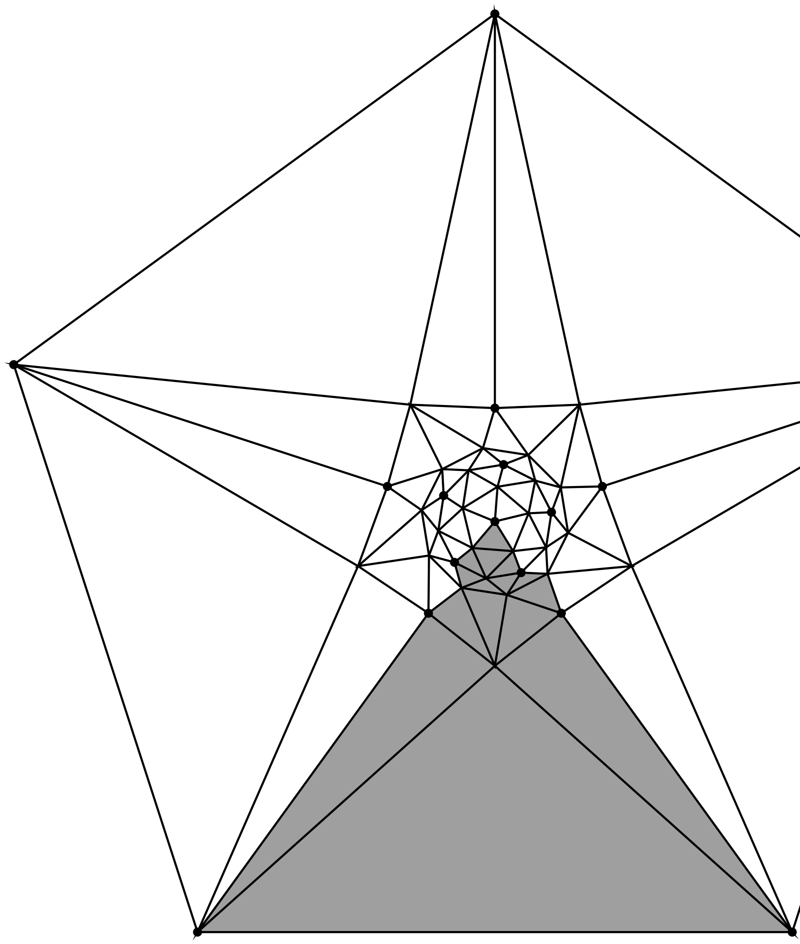




Define height of $p_i := [a, P_{i-1}, P_i]$ for an arbitrary point a.

Minimal pseudotriangulations

Minimal pseudotriangulations (w.r.t. $\subseteq$) are not necessarily minimum-cardinality pseudotriangulations.



A minimal pseudotriangulation has at most $3n - 8$ edges, and this is tight for infinitely many values of n.

[Rote, C. A. Wang, L. Wang, Y. Xu 2003]

Pseudotriangulations in 3-space?

Rigid graphs are not well-understood in 3-space.