Polynomial inequalities representing polyhedra

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joint work with

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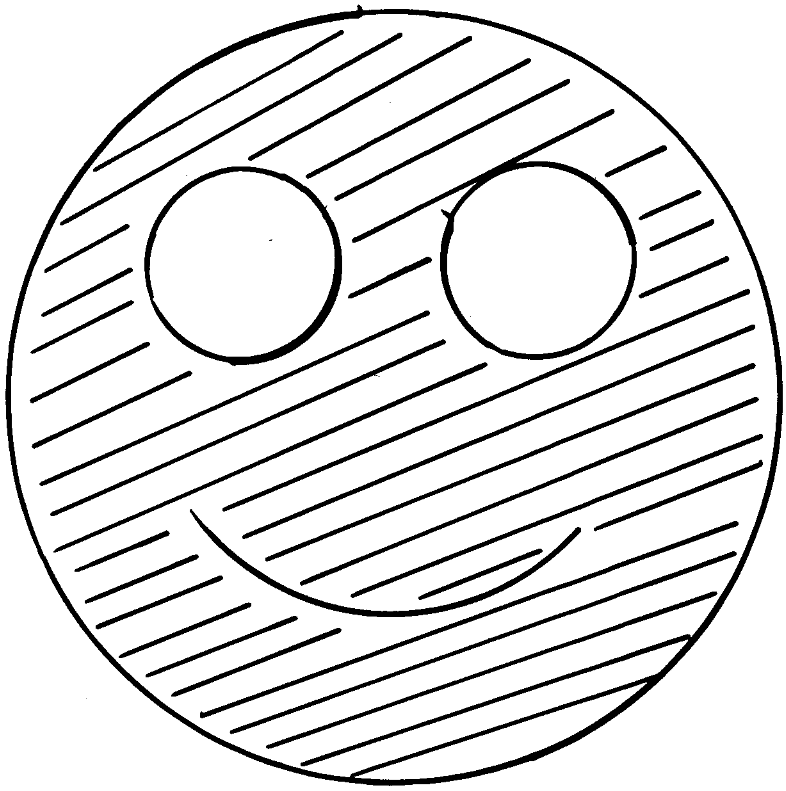
- Let $f_1, \dots, f_\ell, g_1, \dots, g_k \in \mathbb{R}[x]$.
  The finite union of sets of the form

  $$\{ x \in \mathbb{R}^n : f_1(x) \geq 0, \dots, f_\ell(x) \geq 0, \\ g_1(x) > 0, \dots, g_k(x) > 0 \}$$

  is called a semi-algebraic set.

- Semi-algebraic sets are nice

$$S = \{ (x,y) \in \mathbb{R}^2 : \quad x^2+y^2 < 16, \quad (x+2)^2+(y-1)^2 > 1, \; (x-2)^2+(y-1)^2 > 1, \quad (x^2+(y-1)^2 \neq 7 \text{ or } y > -1) \}.$$

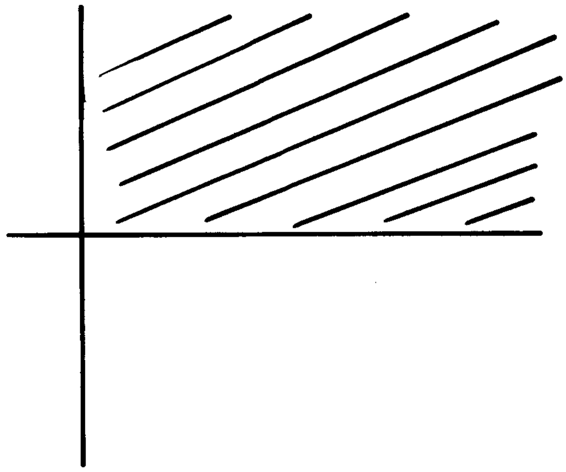


..., but not too nice !

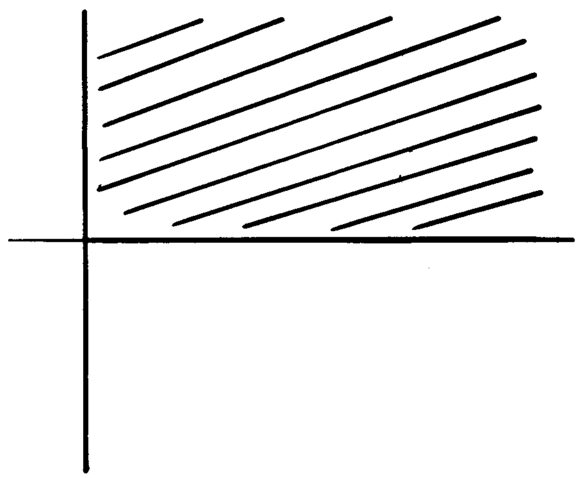
- $\mathrm{cl}(\{x \in \mathbb{R}^n : g_1(x) > 0, \ldots, g_\ell(x) > 0\})$

  $\neq \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_\ell(x) \geq 0\}$

$\{(x,y) \in \mathbb{R}^2 : xy > 0, \ x > 0\}$



$\{(x,y) \in \mathbb{R}^2 : xy \geq 0, \ x \geq 0\}$



- A basic open semi-algebraic set is a set of the form
$$\{x \in \mathbb{R}^n : g_1(x) > 0, \dots, g_\ell(x) > 0\},$$
$g_i \in \mathbb{R}[x]$, $1 \le i \le \ell$.

- A basic closed semi-algebraic set is a set of the form
$$\{x \in \mathbb{R}^n : f_1(x) \geq 0, \dots, f_\ell(x) \geq 0\},$$
$f_i \in \mathbb{R}[x]$, $1 \le i \le \ell$.

- A polyhedron
$$P = \{x \in \mathbb{R}^n : a^i \cdot x \le b_i,\ 1 \le i \le m\},$$
is a basic closed semi-algebraic set.

Theorem [Bröcker (&) Scheiderer, '84, ..., '89]

i) Let $S \subset \mathbb{R}^n$ be a basic open semi-algebraic set. There exist n polynomials $g_1, \dots, g_n \in \mathbb{R}[x]$, s.t.

$$S = \{x \in \mathbb{R}^n : g_1(x) > 0, \dots, g_n(x) > 0\}.$$

ii) Let $S \subset \mathbb{R}^n$ be a basic closed semi-algebraic set. There exist $n(n+1)/2$ polynomials $f_1, \dots, f_{\frac{n(n+1)}{2}} \in \mathbb{R}[x]$, s.t.,

$$S = \{x \in \mathbb{R}^n : f_1(x) \geq 0, \dots, f_{\frac{n(n+1)}{2}}(x) \geq 0\}.$$

Both bounds on the number of needed polynomials are optimal.

- Consequences for polyhedra:

Let

$$P = \{x \in \mathbb{R}^n : a^i \cdot x \leq b_i,\ 1 \leq i \leq m\}$$

be a polyhedron. There exist $n(n+1)/2$ polynomials $f_i \in \mathbb{R}[X]$, s.t.

$$P = \{x \in \mathbb{R}^n : f_1(x) \geq 0, \dots, f_{\frac{n(n+1)}{2}}(x) \geq 0\}.$$

The interior of P can even be described by (at most) n polynomials.

- For $f_i \in \mathbb{R}[x_1, \dots, x_n]$, $1 \le i \le l$, let

  $\mathcal{S}(f_1, \dots, f_l) := \{x \in \mathbb{R}^n : f_1(x) \ge 0, \dots, f_l(x) \ge 0\}$.

- A $\mathcal{S}$-representation of an n-dim. polyhedron

  $P = \{x \in \mathbb{R}^n : a^i \cdot x \le b_i,\ 1 \le i \le m\}$

  consists of l polynomials $f_1, \dots, f_l \in \mathbb{R}[x]$, s.t.,

  $$P = \mathcal{S}(f_1, \dots, f_l).$$

- $P = \mathcal{S}(b_1 - a^1 \cdot x, \dots, b_m - a^m \cdot x)$

- For an n-polyhedron P let $mz(P)$ be the minimum number of polynomials needed in a $\mathcal{S}$-representation of P.

  Let

  $$\overline{mz}(n) = \max\{mz(P) : P \text{ polyhedron}\},$$

  $$mz(n) = \max\{mz(P) : P \text{ polytope}\}.$$

- $n \leq mz(n) \leq \overline{mz}(n) \leq \frac{n(n+1)}{2}$.

- n-cube

$$C^n = \{x \in \mathbb{R}^n : \ -1 \leq x_i \leq 1\}$$

$$= \{x \in \mathbb{R}^n : \ x_i^2 \leq 1\}$$

- n-simplexe

$$T^n = \{x \in \mathbb{R}^n : \ x_i \geq 0, \ \sum x_i \leq 1\}$$

$$= \{x \in \mathbb{R}^n : \ x_i \Big(1 - \sum_{k=i}^{n} x_k\Big) \geq 0, \ 1 \leq i \leq n\}$$

- n-crosspolytope

$$C_*^n = \{x \in \mathbb{R}^n : \ \sum |x_i| \leq 1\}$$

$$= \ ?$$

The 2-dimensional case

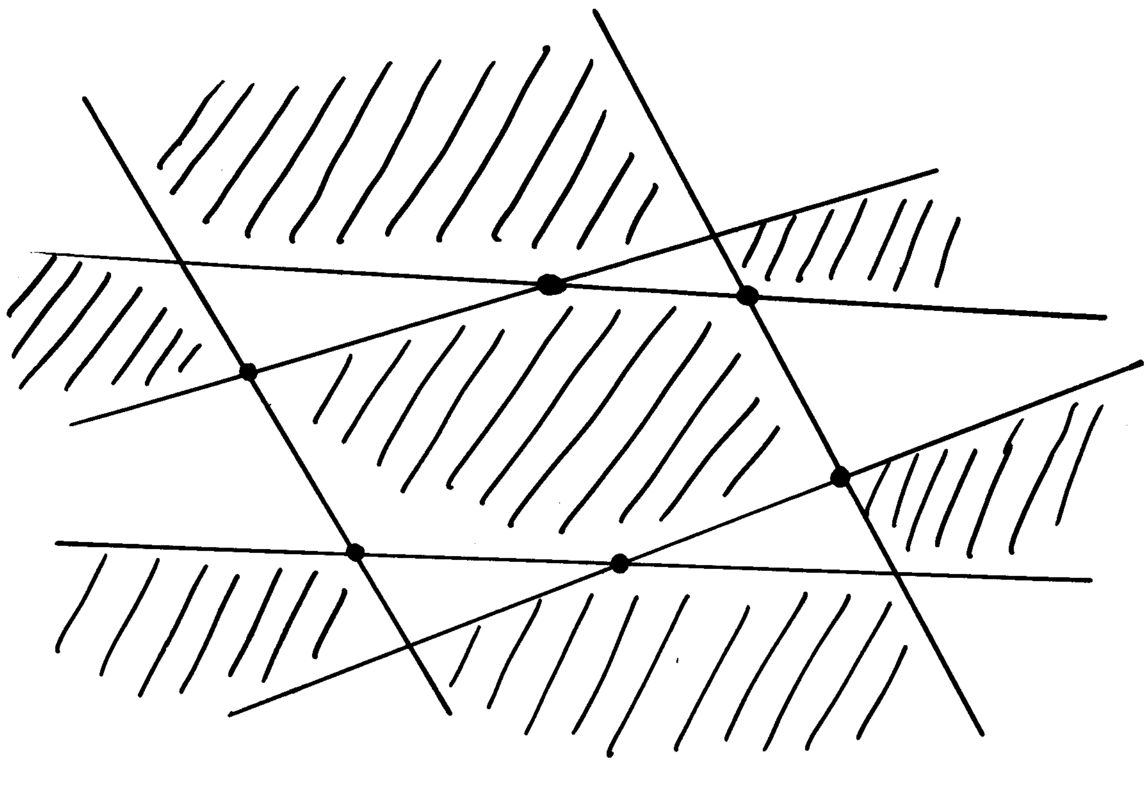
- vom Hofe, 1992:
  Construction of 3 polynomials
  for polygons.

- Bernig, 1998:
  Construction of 2 polynomials
  for polygons $\Rightarrow m_{\mathcal{P}}(2) = 2$.

Idea

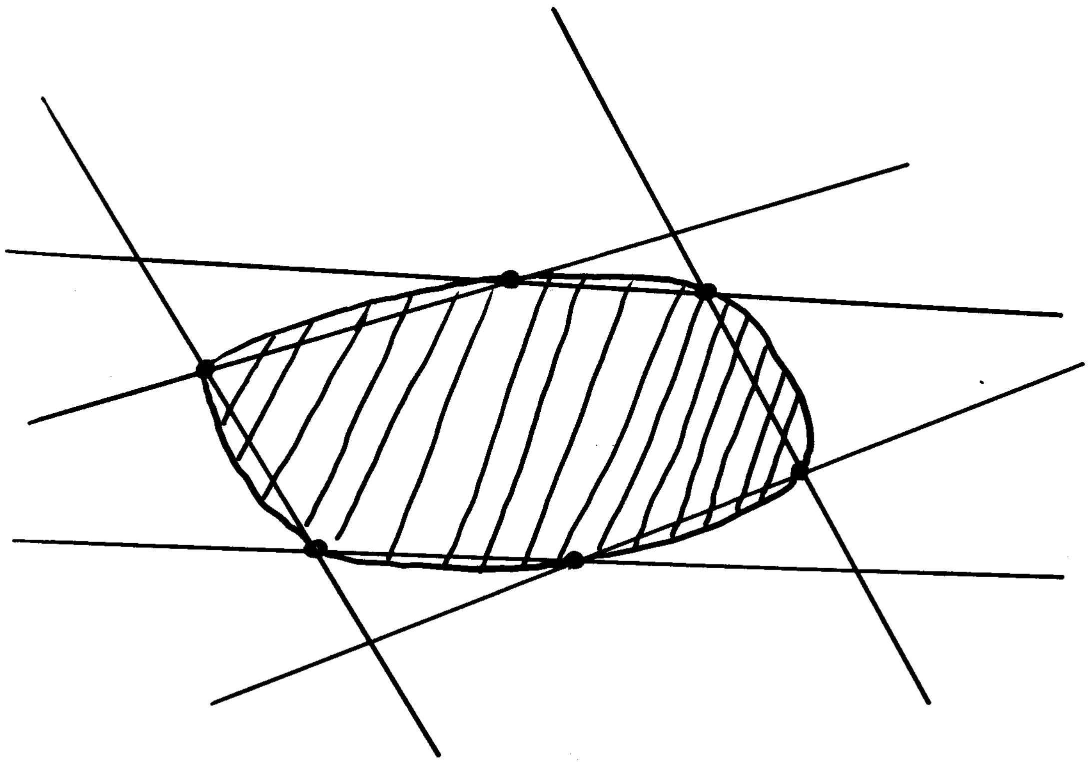
- $P = \{ x \in \mathbb{R}^2 : a^i x \leq b_i, 1 \leq i \leq m \}$

I $f_1(x) = (b_1 - a^1 \cdot x)(b_2 - a^2 \cdot x) \cdot \ldots \cdot (b_m - a^m \cdot x)$



$\{ x \in \mathbb{R}^2 : f_1(x) \geq 0 \}$

II $f_0(x)$ = strictly convex polynomial with $f_0(v) = 1$ for all vertices v of P.



$$\{ x \in \mathbb{R}^2 : f_0(x) \leq 1 \}$$

$$\Rightarrow P = \{ x \in \mathbb{R}^2 : f_1(x) \geq 0,\ 1 - f_0(x) \geq 0 \}$$

- Let $P = \{x \in \mathbb{R}^n : a^i \cdot x \le b_i, 1 \le i \le m\}$ and let $P = \mathcal{S}(f_1, \dots, f_l)$, $f_i \in \mathbb{R}[x]$, $1 \le i \le l$. Then

  i) Each facet defining linear polynomial $b_i - a^i \cdot x$ is a factor of one of the f_is.

  ii) Let F be a k-dimensional face of P. There exist $n-k$ polynomials $f_{s_1}, \dots, f_{s_{n-k}}$, say, s.t.

  $$\mathrm{aff}(F) \subset \{x \in \mathbb{R}^n : f_{s_1}(x) = \dots = f_{s_{n-k}}(x) = 0\}.$$

Corollary: Let $P = \mathcal{P}(f_1, \dots, f_\ell)$.

i) $\sum_{i=1}^{\ell} \deg(f_i) \geq$ # facets of P.

ii) F k-face of P.

$$mp(P) \geq mp(F) + n - k$$

$$(mp(P) \geq n, \ P \text{ polytope})$$

iii) $$mp(n+1) \geq mp(n) + 1$$

$$(\overline{mp}(n+1) \geq \overline{mp}(n) + 1)$$

- Let P be an n-pyramid (n-prism) with basis Q. Then

$$mz(P) = mz(Q) + 1.$$

Corollary:

Every 3-pyramid (3-prism) can be represented by 3 polynomials.

- $mz(n) \le \overline{mz}(n) \le mz(n) + 1.$

• 2002: Let P be a simple n-polytope.

Then $\mu(n) \leq n^n$ polynomials

$f_i \in \mathbb{R}[x]$ can be constructed s.t.

$$P = S(f_1, \dots, f_{\mu(n)}).$$

In particular, we can take

$\mu(2) = 3$ and $\mu(3) = 6$.

- 2003:
  - Let $C = \{x \in \mathbb{R}^n : a^i x \leq 0,\ 1 \leq i \leq m\}$ be a pointed cone. Then $2n-2$ polynomials $f_j \in \mathbb{R}[X]$ can be constructed s.t.

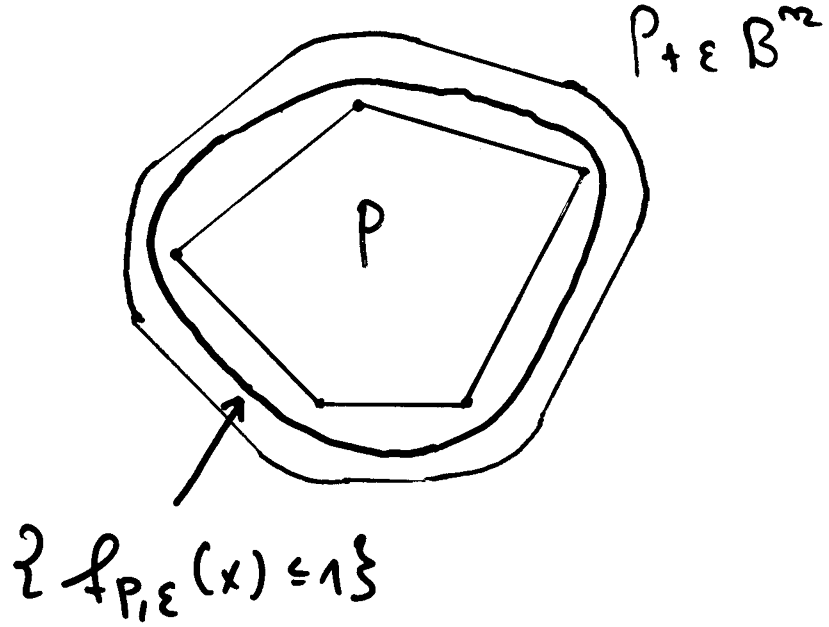
    $$C = \mathcal{S}(f_1, \dots, f_{2n-2}).$$

  - For a polytope $P \subset \mathbb{R}^n$ we can construct $2n-1$ polynomials $f_j \in \mathbb{R}[X]$ s.t.

    $$P = \mathcal{S}(f_1, \dots, f_{2n-1}).$$

- Lemma: Let P be an n-polytope and let $\varepsilon > 0$. There we can construct a (strictly convex) polynomial $f_{P,\varepsilon}(x)$ s.t.

$$P \subset \{x \in \mathbb{R}^n : f_{P,\varepsilon}(x) \leq 1\} \subset P + \varepsilon B^n.$$



- Lemma: Let C be an n-dim. pointed cone and let u_0 be an outer unit normal vector of the vertex 0.

Let $f_0 = -u_0 \cdot x$ and let $\varepsilon > 0$. Then we can construct a polynomial $f_{C,\varepsilon}$ s.t.

$$\{x + \varepsilon (u_0 \cdot x) B^n : x \in C\} \subset S(f_{C,\varepsilon}, f_0)$$
$$\subset \{x + \omega_C \cdot \varepsilon (u_0 \cdot x) B^n : x \in C\},$$

where ω_C is a constant depending only on C.

Furthermore, we have

$$\{x \in \mathbb{R}^n : f_0(x) = 0, f_{C,\varepsilon}(x) = 0\} = \{0\}$$

• Corollary: Let $C=\{x\in\mathbb{R}^n: a_i x\le 0\}$ be an n-dim. pointed cone and let F be a k-face. Let

$I_F=\{i: a_i x=0 \ \forall x\in F\}$ and let

$C_F=\{x\in\mathbb{R}^n: a_i x\le 0,\ i\in I_F\}$.

Let u_F be an outer unit normal vector of the face F. Let

$f_F=-u_F\cdot x$ and let $\varepsilon>0$. Then we can construct a polynomial $f_{C_F,\varepsilon}$ s.t.

$$\{x+\varepsilon(u_F\cdot x)B^n: x\in C_F\}\subset S(f_{C_F,\varepsilon}, f_F)$$
$$\subset\{x+w_{C_F}\varepsilon(u_F\cdot x)B^n: x\in C_F\},$$

where w_{C_F} is a constant depending only on C.

Furthermore, we have

$$\{x\in\mathbb{R}^n: f_F(x)=0,\ f_{C_F,\varepsilon}(x)=0\}=\operatorname{lin}F.$$

- Let G be an n-dim. pointed cone and let $\tilde{\mathfrak{F}}_k$ be the set of all k-faces of G. Let $\varepsilon_i > 0$, $0 \le i \le n-1$.

For $0 \le k \le n-1$ let

$$h_{k,1} = \prod_{F \in \tilde{\mathfrak{F}}_k} f_F \quad \text{and} \quad h_{k,2} = \prod_{F \in \tilde{\mathfrak{F}}_k} f_{G_F, \varepsilon_k}.$$

- We can determine numbers $\varepsilon_0, \dots, \varepsilon_{n-1}$ s.t.

$$G = \{ x \in \mathbb{R}^n : h_{k,1}(x) \ge 0,\ h_{k,2}(x) \ge 0,\ 0 \le k \le n-1 \}$$