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Kähler manifolds with positive spectrum

(Joint with Peter Li)

Recall: Laplacian comparison

$Ric_{M^n} \geq (n-1)K \Rightarrow \Delta_M r \leq \Delta_{M_K} r$

M_K = space form of constant curvature K.

S.Y. Cheng: $\lambda_0(M) \leq \lambda_0(M_K)$

bottom spectrum

In particular,

$Ric_{M^n} \geq -(n-1) \Rightarrow \lambda_0(M) \leq \frac{(n-1)^2}{4} = \lambda_0(\mathbb{H}^n)$

Q: equality?

Thm (Li-W): Assume $Ric_{M^n} \geq -(n-1)$ and $\lambda_0(M) = \frac{(n-1)^2}{4}$. Then

$n \geq 4$, $H_{n-1}(M, \mathbb{Z}) = \{0\}$ or

$M = \mathbb{R} \times N$ (compact) with

$$ds_M^2 = dt^2 + e^{2t} ds_N^2$$

$n=3$, $H_{n-1}(M, \mathbb{Z}) = 0$ or

$M = \mathbb{R} \times N$ (compact) with

$$ds_M^2 = dt^2 + e^{2t} ds_N^2 \quad \text{or} \quad ds_M^2 = dt^2 + \cosh^2 t \, ds_N^2$$

$n=2$, not true!

But one does know that $\exists$ at most one cusp.

Concerning the infinite volume ends, we can relax the assumption on $\lambda_0(M)$.

Thm (Li-W): Assume $Ric_{M^n} \geq -(n-1)$, $\lambda_0(M) \geq n-2$ and $Vol(B_x(1)) \geq C$, $\forall x \in M$. Then for $n \geq 3$, $H_{n-1}(M, \mathbb{Z}) = 0$ or

$M = \mathbb{R} \times N^{n-1}$ (compact) with $ds_M^2 = dt^2 + \cosh^2 t \, ds_N^2$.

Rk: For conformally compact M, the result is due to X. Wang. His result generalized earlier work of Witten-Yau, Cai-Galloway.

Turn to Kähler case.

Consider (M^m, ω) Kähler,

$\omega =$ Kähler form, $m = \dim_{\mathbb{C}} M$.

Bisectional curvature $BK_M \geq K \overset{\text{def}}{\iff}$

$R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq K(1 + g_{\alpha\bar{\beta}})$ under

unitary frame.

Model: $K = -1, 0, 1$

$\mathbb{C}H^m, \mathbb{C}^m, \mathbb{C}P^m$

Note that: On $\mathbb{C}H^m$, $Ric_{\mathbb{C}H^m} = -2(m+1)$

$\Delta_{\mathbb{C}H^m} \ln \cosh r = 2m$.

So $\Delta r \leq 2m$ for $r \gg 1$

But the Laplacian comparison we mentioned earlier $\Rightarrow$ $\Delta r \leq \sqrt{4m^2+2m-2}$, $r \gg 1$.

Not sharp!

Thm (Li-W): $\Delta_M r \leq \Delta_{M_K} r$ if

M Kähler with $BK_M \geq K$, where M_K is the complex space form.

Just like S.Y. Cheng case,

$\lambda_0(M) \leq \lambda_0(M_K)$ if $BK_M \geq K$.

So $\lambda_0(M^m) \leq m^2$ if $BK_{M^m} \geq -1$.

as $\lambda_0(\mathbb{C}H^m) = m^2$.

Again, one asks what to say about M if $BK_{M^m} \geq -1$ and $\lambda_0(M^m) = m^2$.

Thm (Li-W): If $BK_{M^m} \geq -1$, $\lambda_0(M) \geq m$ and $vol(B_x(1)) \geq c$ for all $x \in M$, then

$H_{2m-1}(M, \mathbb{Z}) = 0$ for $m \geq 2$.

For the proof, we inspect the number of ends. The conclusion on homology follows then by a simple topological argument.

Li-Tam theory $\Rightarrow$ For any complete manifold M, number of ends of M $= \dim \mathcal{H}$ $\rightsquigarrow$ a space of harmonic functions constructed explicitly.

Specifically, each end $\rightsquigarrow$ barrier function f s.t. $f|_{\partial E} = 0$, $\Delta f = 0$ on E

and $f > 0$ in E.

Each pair of ends $\rightsquigarrow$ global function

u s.t. $\Delta u = 0$ on M.

Crucial fact: (P. Li)

M Kähler, $\int_M |\nabla u|^2 < \infty \Rightarrow u$ pluriharmonic

Also, if pair of ends with infinite volume,

then for $u \in \mathcal{H}$, $|\nabla u|^2$ decays

exponentially $(\approx e^{-2\sqrt{\lambda_0}\, r})$.

The proof can be finished by squeezing

the Bochner formula.

The preceding argument no longer works

if we were to deal with cusps.

The harmonic functions have infinite energy. So it is unclear whether they are pluriharmonic or not.

We have the following partial result:

<u>Thm</u> (Li-W): M^2 complete Kähler with $BK_M \geq -1$ and $\lambda_0(M) = 4$. Then M has at most 3 cusps.

<u>RK</u>: ① We doubt the estimate is sharp.

② For $m > 2$, no result yet.

The proof relies on the following computation

$$\Delta |u_{\alpha\bar{\beta}}|^{\frac{m-1}{m}} \geq -4(m-1) |u_{\alpha\bar{\beta}}|^{\frac{m-1}{m}}$$

for harmonic function u.

Also, for $u \in \mathcal{H}$ constructed by Li-Tam theory, we have the sharp estimates

$$\begin{cases} u(x) \le c\, e^{2m\, r(x)} \\ \mathrm{Vol}(B_x(1)) \le c\, e^{-2m\, r(x)} \end{cases} \quad \text{along cusps}$$

$$\begin{cases} u(x) \le c\, e^{-2m\, r(x)} \\ \mathrm{Vol}(B_x(r)) \le c\, e^{2m\, r(x)} \end{cases} \quad \text{along ends}$$

with infinite volume.

Putting together and specializing to $m=2$, we get

$$\Delta |u_{\alpha\bar\beta}|^{\frac{1}{2}} = -4\, |u_{\alpha\bar\beta}|^{\frac{1}{2}}.$$

The result then follows.