THE AMOEBA OF A DISCRIMINANT

MIKAEL PASSARE, TIMUR SADYKOV AND AUGUST TSIKH

FEBRUARY 23, 2004

DEFINITION. Let $A \subset \mathbf{Z}^n$ be finite and f be $A-$supported polynomial. $A-$*discriminant* is the polynomial in the coefficients of f which equals zero if f and grad$f$ vanish simultaneously.

EXAMPLE. $A = \{0, 1, 2\} \subset \mathbf{Z}$,
$f = ay^2 + by + c$.
$A-$discriminant: $b^2 - 4ac$.

EXAMPLE. For

$$A = \{(0,0), (1,0), \ldots, (m,0),\\ (0,1), (1,1), \ldots, (n,1)\} \subset \mathbf{Z}^2,$$

the corresponding $A-$discriminant is the resultant of two univariate polynomials.

EXAMPLE. $A-$discriminant of a bilinear form $\sum a_{ij}x_iy_j$ is the determinant of the matrix (a_{ij}).

DEFINITION. Let f be a Laurent polynomial

$$f = \sum_{\alpha} c_{\alpha_1,\ldots,\alpha_n}\, x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

Its *amoeba* $\mathcal{A}_f$ is defined to be the image of the hypersurface $\{f = 0\}$ under the mapping

$$\mathrm{Log} : (x_1, \ldots, x_n) \mapsto (\log|x_1|, \ldots, \log|x_n|).$$

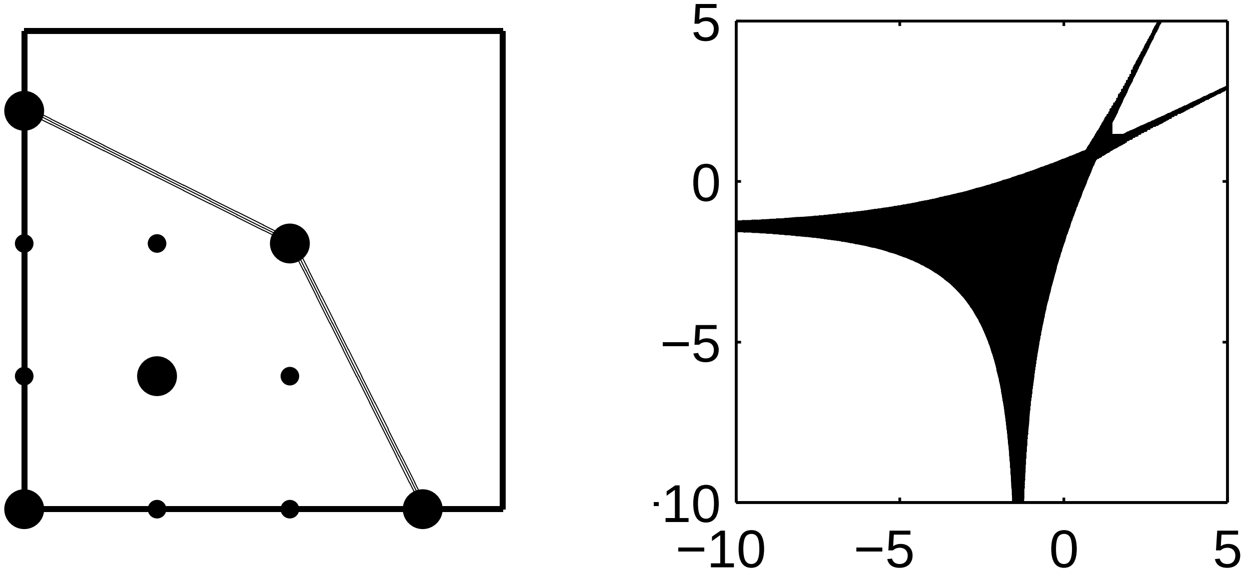
EXAMPLE. The discriminant of the polynomial

$$y^3 + x_1\, y^2 + x_2\, y - 1$$

is given by

$$x_1^2 x_2^2 + 4x_1^3 - 4x_2^3 - 18x_1x_2 - 27. \quad (1)$$

The Newton polytope and the amoeba of (1):



Connected components of the amoeba complement are convex.

PROBLEM. How to describe the zero locus of an A−discriminant?

THEOREM. (Kapranov, 1991) A−discriminantal hypersurface is birationally equivalent to the projective space.

THEOREM. (Gelfand, Kapranov, Zelevinsky, 1994) The Newton polytope of the discriminant of a univariate polynomial is combinatorially equivalent to a cube.

THEOREM. (Forsberg, Passare, Tsikh, 2000)

$$\#\,\text{vertices of } \mathcal{N}_f \;\le\; \#\,{}^c\!\mathcal{A}_f \;\le\; \#\,\mathcal{N}_f \cap \mathbf{Z}^n.$$

DEFINITION. A polynomial (an amoeba, a hypersurface) is called *solid*, if the lower bound is attained.

THEOREM. A**-discriminants have solid amoebas.**

DEFINITION. A function is called *hypergeometric* if it satisfies a regular holonomic system of the form

$$x_i P_i(\theta) - Q_i(\theta), \quad i = 1, \ldots, n, \tag{2}$$

where P_i and Q_i are nonzero polynomials and

$$\theta = \left(x_1 \frac{\partial}{\partial x_1}, \ldots, x_n \frac{\partial}{\partial x_n} \right).$$

Let $J = (x_1 P_1(\theta) - Q_1(\theta), \ldots, x_n P_n(\theta) - Q_n(\theta))$,

char $(J) =$

$\{(x, z) \in \mathrm{C}^{2n} : \sigma(P)(x, z) = 0, \forall P \in J\}.$

THEOREM. (Bernstein, 1972) The dimension of the characteristic variety of a system in n variables is $\geq n$.

Holonomic: The dimension of the characteristic variety of (2) equals n.

THEOREM (Dickenstein, Matusevich, Sadykov, 2003). A bivariate hypergeometric system is generically holonomic.

Regular: **No torsion + moderate growth of solutions in a neighbourhood of a singularity in $\mathbf{P}^n$.**

EXAMPLE. The system of equations

$$x_1\theta_1(\theta_1+\theta_2)-(\theta_1+1)(\theta_1+\theta_2),$$

$$x_2\theta_2(\theta_1+\theta_2)-(\theta_2+1)(\theta_1+\theta_2)$$

is not regular holonomic. Any function on the projective line is a solution to it.

EXAMPLE.

The system of differential equations:

$$x(1-x)\partial_x^2 - xy\partial_x\partial_y + (c-(a+b+1)x)\partial_x - by\partial_y - ab,$$

$$y(1-y)\partial_y^2 - xy\partial_x\partial_y + (c'-(a+b'+1)y)\partial_y - b'x\partial_x - ab'$$

is regular holonomic for generic parameters.

The singular locus of this system:

$$S = \{xy(1-x)(1-y)(1-x-y) = 0\}.$$

THEOREM.

A basis in the solution space of a hypergeometric system with commuting operators and generic parameters has the form

$$y_I(x) = \sum_{k \in \mathbb{N}^n} \varphi(k)(tx)^{k+\gamma_I},$$

where

$$\varphi(k) = \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, k + \gamma_I \rangle + c_i)}{\prod_{\nu=1}^{n} \prod_{j=1}^{d_\nu} \Gamma(k_\nu + u_{\nu j} + 1)}.$$

THEOREM. Singularities of hypergeometric functions are algebraic and solid.

EXAMPLE.

$$f = \sum_{k_1,k_2\geq 0} \frac{\Gamma(k_1+k_2+1)}{\Gamma(k_1+1)\Gamma(k_2+1)} x_1^{k_1} x_2^{k_2}$$

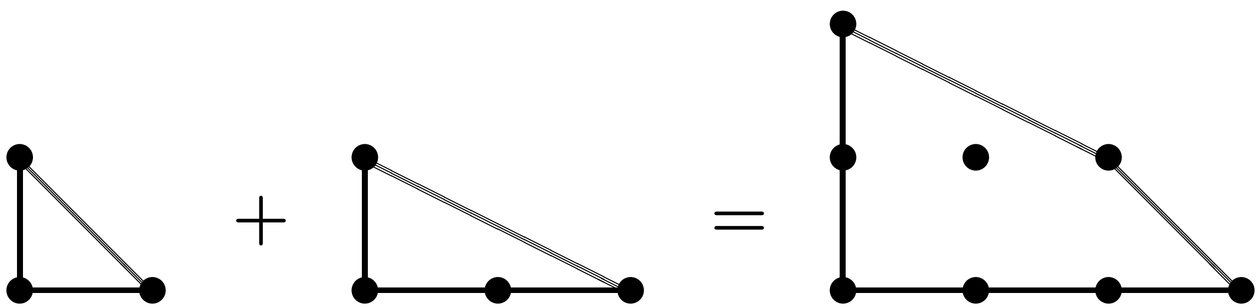
$$= \frac{1}{1-x_1-x_2}$$

$$g = \sum_{k_1,k_2\geq 0} \frac{\Gamma(k_1+2k_2+2)}{\Gamma(k_1+1)\Gamma(2k_2+2)} x_1^{k_1} x_2^{k_2}$$

$$= \frac{1}{(1-x_1)^2-x_2}$$

$$f \odot g = \sum_{k_1,k_2\geq 0} \frac{\Gamma(k_1+k_2+1)\Gamma(k_1+2k_2+2)}{\Gamma^2(k_1+1)\Gamma(k_2+1)\Gamma(2k_2+2)} x_1^{k_1} x_2^{k_2}$$

$$4x_1^3 - x_1^2 x_2 - 12x_1^2 + 20x_1x_2 - 4x_2^2$$
$$+12x_1 + 8x_2 - 4$$



THEOREM.

The Hadamard product of double non-confluent hypergeometric series corresponds to the Minkowski sum of the Newton polytopes of the polynomials which define their singularities.

REMARKS.

1. Not every solid polynomial determines the singularity of a hypergeometric function.

2. In two variables, any convex integer polygon is the Newton polytope of some polynomial which defines the singularity of a hypergeometric function.

3. Discriminants of univariate polynomials have solid amoebas.