The Bergman complex of a matroid and phylogenetic trees

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Program

1. Amoebas and the Bergman complex.
2. The Bergman complex of a linear space.
3. The connection with phylogenetic trees.
4. The results.

1. Amoebas and the Bergman complex

Consider a variety $X \subset \mathbb{C}^n$, described by a system of polynomial equations in $\mathbb{C}[z_1, \ldots, z_n]$:

$$f_1(z_1, \ldots, z_n) = \cdots = f_k(z_1, \ldots, z_n) = 0.$$

Question: Given $r_1, \ldots, r_n > 0$, is there a solution $z \in X$ with

$$|z_1| = r_1, \ldots, |z_n| = r_n?$$

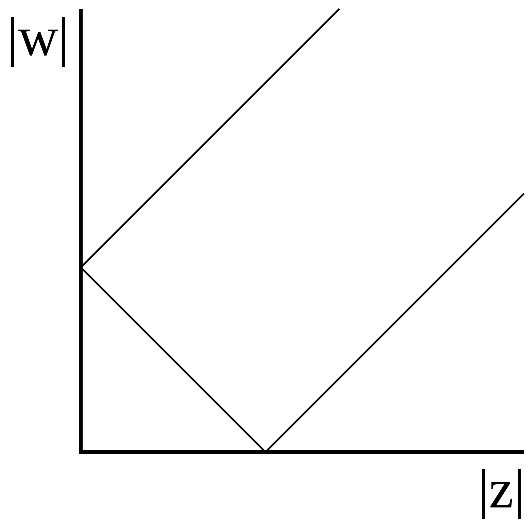
The amoeba of X is

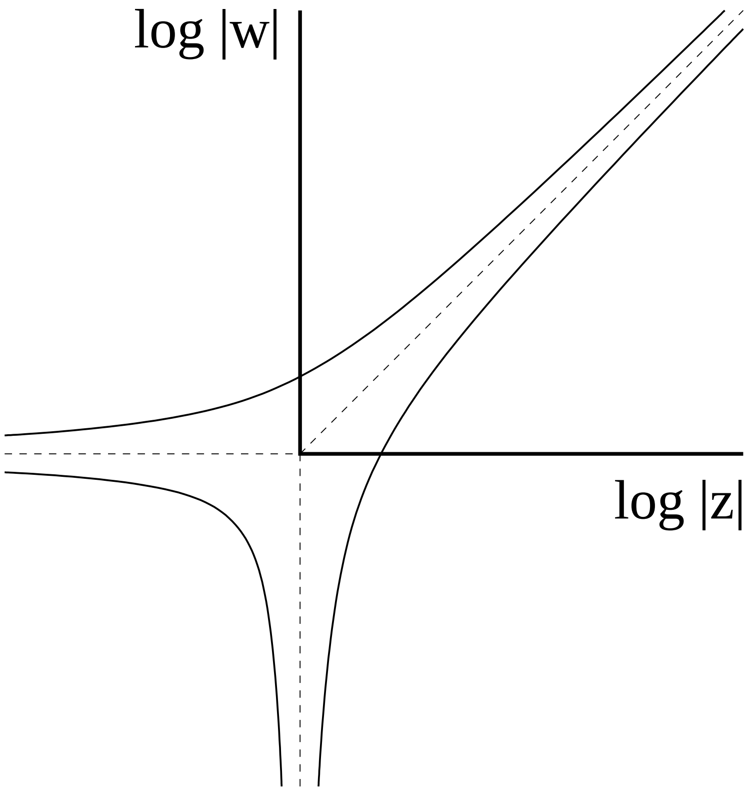
$$\mathcal{A}(X) = \text{Log } X = \{(\log|z_1|, \ldots, \log|z_n|) \ : \ z \in X \cap (\mathbb{C}^*)^n\}.$$

Example: $X = \{(w,z) \in \mathbb{C}^2 \mid 1+w+z=0\}$

There is a solution with given $|w|$ and $|z|$ if and only if

$$1 \leq |w|+|z|, \quad |w| \leq 1+|z|, \quad |z| \leq 1+|w|.$$





In general, amoebas are very difficult to describe. Many open problems! Their "tentacles" are simpler.

The Bergman complex of X, $\mathcal{B}(X)$, is a subset of the sphere S^{n-1}. It is (roughly) the set of directions where $\mathcal{A}(X)$ goes to infinity.

The Bergman fan, $\widetilde{\mathcal{B}}(X)$, is the fan over the Bergman complex.

Theorem. (Bergman, '71; Bieri and Groves, '84)

If X is d-dimensional and irreducible, then $\mathcal{B}(X)$ is a pure $(d-1)$-dimensional polyhedral complex.

Let I be the ideal of X.

Let $\text{in}_\omega(I)$ be the initial ideal w.r.t. $\omega \in \mathbb{R}^n$:

$$\text{in}_{(0,2,1)}(2xy^2 - x^3z + 3z^4) = 2xy^2 + 3z^4$$

0+4 0+1 4

$$\text{in}_\omega(I) = < \text{in}_\omega(f) \,|\, f \in I >$$

Theorem. (Sturmfels, '02)
$\mathcal{B}(X) = \{\omega \in S^{n-1} \mid \text{in}_\omega(I) \text{ contains no monomials}\}$

This gives another proof that $\mathcal{B}(X)$ is a pure polyhedral complex.

2. The Bergman complex of a linear space.

Fix a coordinate system of $\mathbb{C}^n$, and let V be a subspace.

Think: ω_i is the weight of coordinate x_i.

If V satisfies a *minimal* equation $a_1 x_{i_1} + \cdots + a_k x_{i_k} = 0$, say that the set of variables $\{x_{i_1}, \ldots, x_{i_k}\}$ is a circuit.

Corollary. The Bergman complex of a subspace V of $\mathbb{C}^n$ is $\{\omega \in S^{n-2}$: every circuit achieves its ω-max more than once.$\}$

(Here $S^{n-2} = \left\{\omega \in \mathbb{R}^n \mid \sum \omega_i^2 = 1, \sum \omega_i = 0\right\}$.)

Goal: Describe $\mathcal{B}(V)$ topologically, combinatorially.

Example. $V = \{x \in \mathbb{C}^4 \mid x_1 + x_2 + x_3 = 0\,,\, x_3 = x_4\}$

A vector ω is in the Bergman complex iff:

- $\max\{\omega_1, \omega_2, \omega_3\}$ is achieved twice, *and*
- $\max\{\omega_3, \omega_4\}$ is achieved twice.

This happens iff

- $\omega_3 = \omega_4 = \omega_2 \geq \omega_1$, *or*
- $\omega_3 = \omega_4 = \omega_1 \geq \omega_2$, *or*
- $\omega_1 = \omega_2 \geq \omega_3 = \omega_4$.

$$\mathcal{B}(V) = \left\{ (-\tfrac{\sqrt{3}}{2}, \tfrac{\sqrt{3}}{6}, \tfrac{\sqrt{3}}{6}, \tfrac{\sqrt{3}}{6},), (\tfrac{\sqrt{3}}{6}, -\tfrac{\sqrt{3}}{2}, \tfrac{\sqrt{3}}{6}, \tfrac{\sqrt{3}}{6}), (\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}) \right\}.$$

3. The connection with phylogenetic trees.

Let $\mathbb{C}^{\binom{n}{2}} = \{(x_{12}, x_{13}, x_{23}, \ldots, x_{n-1,n}) : x_{ij} \in \mathbb{C}\}$.

$K_n = \{x \in \mathbb{C}^{\binom{n}{2}} \mid x_{ij} + x_{jk} = x_{ik}$ for $i < j < k\}$.

Proposition. (A., Klivans)

For $\omega \in \mathbb{R}^{\binom{n}{2}}$, the following are equivalent:

1. $\omega \in \widetilde{\mathcal{B}}(K_n)$.
2. For all $i < j < k$, $\max\{\omega_{ij}, \omega_{jk}, \omega_{ik}\}$ is achieved twice.

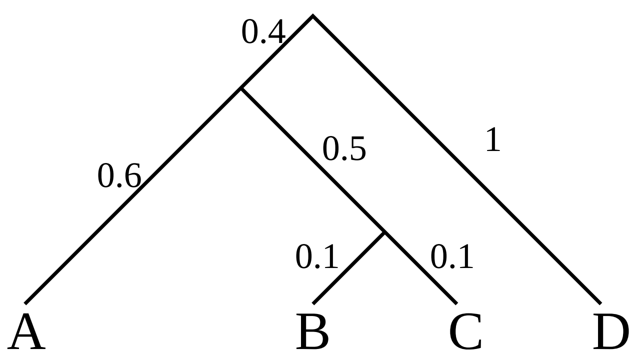
Call such a ω an ultrametric.

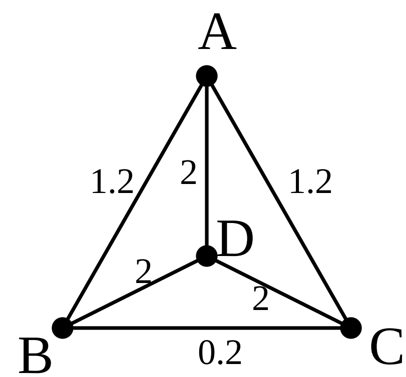
(Think: A weighting of the edges of the complete graph K_n, such that in each triangle, the two heaviest edges have equal weights.)

One source of ultrametrics:

T = equidistant n-tree (rooted metric tree with n labelled leaves, and all distances from the root to the leaves equal to 1.)

d_{ij} = distance between leaves i and j





Theorem. (Semple and Steel, 2003)
A vector $\delta \in \mathbb{R}^{\binom{n}{2}}$ is an ultrametric if and only if it is the distance function of an equidistant n-tree.

Therefore, we can think of the Bergman fan $\widetilde{\mathcal{B}}(K_n)$ as a space of phylogenetic trees.

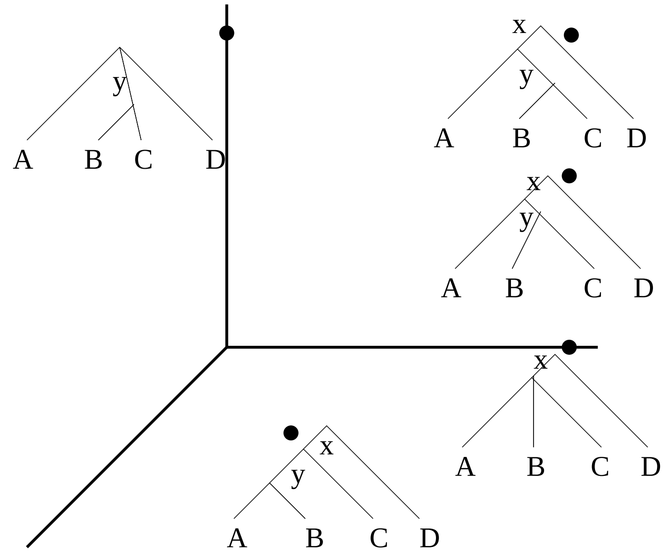
The space of phylogenetic trees $\mathcal{T}_n$.

(Vogtmann, 1990; Whitehouse, 1996; Billera, Holmes, V., 2001)

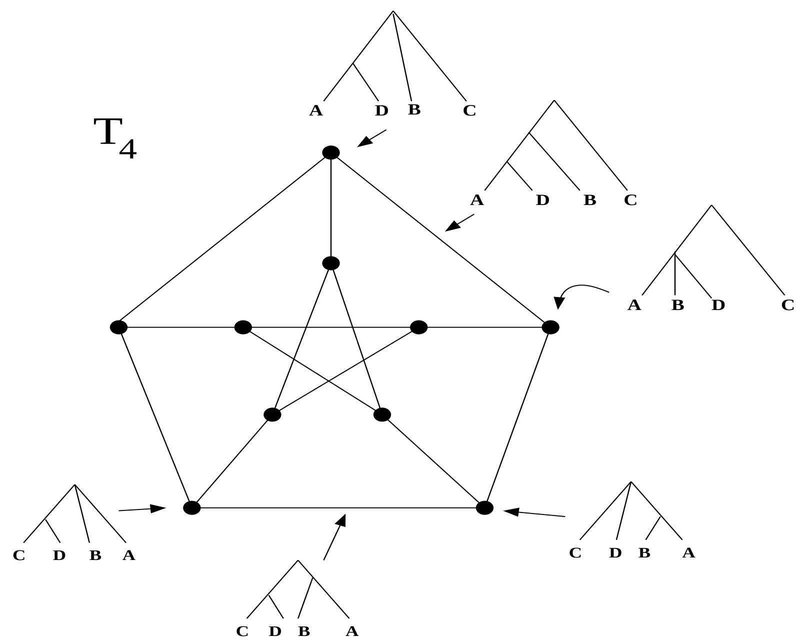
A binary n-tree T has $n-2$ internal edges. An orthant $\mathbb{R}^{n-2}_{\geq 0}$ parameterizes the possible equidistant n-trees of that shape.

When some edge lengths are 0, we get "degenerate" non-binary trees, which could come from different binary trees.

Glue the $(n-2)$-dimensional orthants where they agree.



Whitehouse complex T_n = link of the origin in $\mathcal{T}_n$



Theorem. (Vogtmann, 1990; Robinson and Whitehouse, 1996; Trappmann and Ziegler, 1998; Wachs, 1998; Sundaram, 1999)

T_n is a simplicial complex, homotopy equivalent to a wedge of $(n-1)!$ $(n-3)$-dimensional spheres.

Fairly complicated proofs: shellability, Quillen's fiber lemma, ...

We have two different parameterizations of equidistant n-trees.

- $\mathcal{T}_n$: combinatorial type, internal edge lengths.
- $\widetilde{\mathcal{B}}(K_n)$: distances between leaves.

We get a map $f : \mathcal{T}_n \to \widetilde{\mathcal{B}}(K_n)$.

Theorem. (A., Klivans, 2003)

The map f is a piecewise linear homeomorphism between the Bergman fan $\widetilde{\mathcal{B}}(K_n)$ and the space of phylogenetic trees $\mathcal{T}_n$.

So if we are able to describe $\widetilde{\mathcal{B}}(K_n)$, we get a description of $\mathcal{T}_n$ for free.

4. The theorems.

The short version:

Theorem. (A., Klivans)

The Bergman complex of V is homotopy equivalent to a wedge of $\hat{\mu}(L_V)$ $(\dim V - 2)$-dimensional spheres.

The number $\hat{\mu}(L_V)$ is very well-understood in enumerative combinatorics.

Corollary.

The Whitehouse complex T_n is homotopy equivalent to a wedge of $(n-1)!$ $(n-3)$-dimensional spheres.

The details:

We need the order complex of the lattice of flats of a matroid.

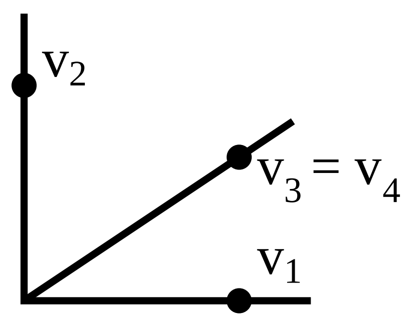
Example. $V = \{x \in \mathbb{C}^4 \mid x_3 = x_4\}$ Circuit: 34.

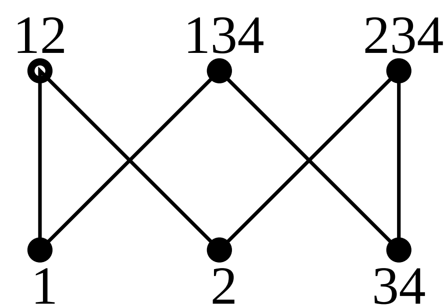
A set F is a flat if $|F - C| \neq 1$ for all circuits C.

Flats of V: $\emptyset, 1, 2, 34, 12, 134, 234, 1234$.

(F cannot contain all but one element of C.)

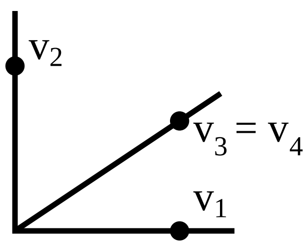
The lattice of flats L_V is the poset of flats ordered by containment. It is a lattice. Let $\bar{L}_V = L_V \backslash \{\hat{0}, \hat{1}\}$.

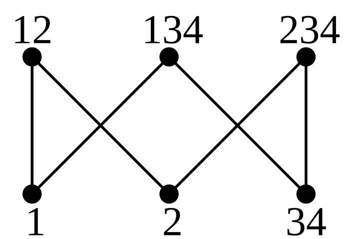


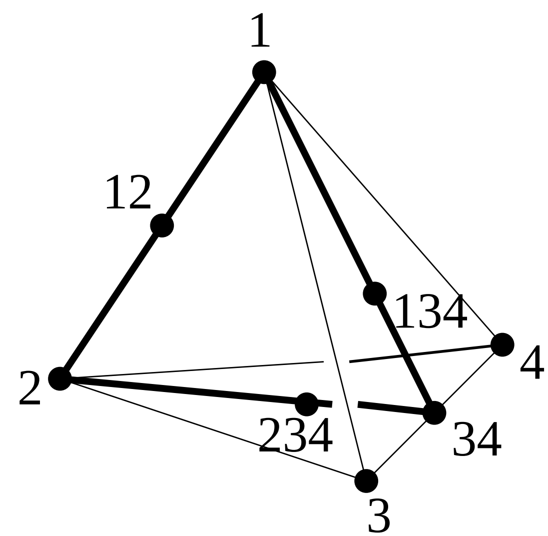


The order complex $\Delta(\bar{L}_V)$ of $\bar{L}_V$ is the following simplicial complex:

- vertices: elements of $\bar{L}_V$
- faces: chains of $\bar{L}_V$







Theorem. (Björner, 1992)

$\Delta(\bar{L}_V)$ is a pure, shellable simplicial complex. It is homotopy equivalent to a wedge of $\hat{\mu}(L_V)$ $(\dim V - 2)$-dimensional spheres.

Theorem. (A. and Klivans, 2003)

(Let M be a loopless matroid.)

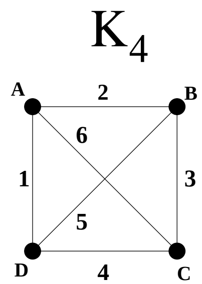
(A subdivision of) the Bergman complex of M is (a geometric realization of) $\Delta(\bar{L}_M)$.

When $M = K_n$,

- The Bergman complex $\mathcal{B}(K_n)$ is homeomorphic to the Whitehouse complex.
- $L_{K_n} = \Pi_n$, the partition lattice.
- $\Delta(\bar{\Pi}_n)$ is a wedge of $(n-1)!$ $(n-3)$-dimensional spheres.

Corollary.

(A subdivision of) the Whitehouse complex is (a geometric realization of) $\Delta(\bar{\Pi}_n)$.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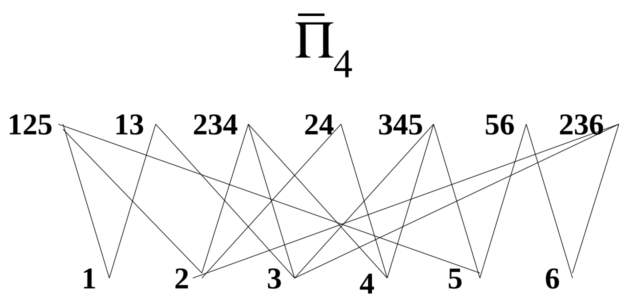
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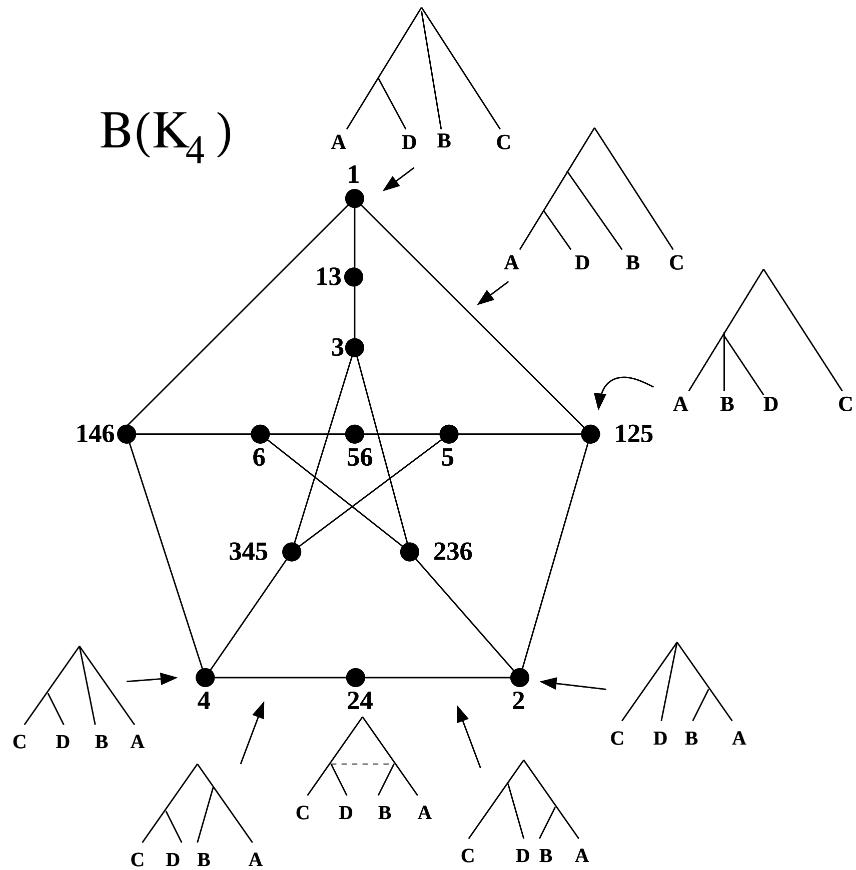
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C D B A
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Thank you.

The preprint is available at:

- www.msri.org/∼federico
- arxiv:math.CO/0311370