Kähler Ricci flow on compact and complete noncompact manifolds

Lei Ni
University of California, San Diego

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1) Some backgrounds in Kaehler-Ricci flow geometry.

The Kaehler-Ricci flow equation:

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = -R_{\alpha\bar{\beta}}(x,t). \qquad (\textbf{KRF})$$

Basic property: It stays in Kaehler category.

When manifold is compact, one also study the normalized KRF:

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = g_{\alpha\bar{\beta}}(x,t) - R_{\alpha\bar{\beta}}(x,t). \qquad (\textbf{NKRF})$$

Special case: $c_1(M) = [\omega]$. **NKRF** preserves the Kaehler class of the initial metric. Here ω is the Kaehler form of $g_{\alpha\bar{\beta}}(x,0)$. Then the Kaehler form of the metric $g_{\alpha\bar{\beta}}(x,t)$ can be written as $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi(x,t)$ for some function $\phi(x,t)$.

Since the deformed metric can be expressed in terms of the Hessian of function (real valued) we can reduce **NKRF** to a single Monge-Ampere equation.

Let f be the potential function satisfying $\sqrt{-1}\partial\bar{\partial}f = Ric(\omega) - \omega$ in the case $c_1(M) = [\omega]$. $g_{\alpha\bar{\beta}}(x,t) = g_{\alpha\bar{\beta}}(x,0) + \phi_{\alpha\bar{\beta}}(x,t)$ with $\phi(x,0) = 0$. Then the **NKRF** flow reduces to

$$\frac{\partial\phi}{\partial t} = \log\frac{\omega_{\phi}^{m}}{\omega^{m}} + \phi - f. \quad (1)$$

This simplifies things quite bit.

The similar reduction can be done if $c_1(M) = k[\omega]$.

2. Existence

a) Compact case:

Short time existence: Hamilton. In case $c_1(M) = k[\omega]$ follows from standard PDE theory, due to the reduction mentioned before.

Long time existence:

Theorem (Cao) In the case $c_1(M) = [\omega]$, the NKRF has lone time existence.

C^0-estimate is easy: If $v = \phi_t$.

$$v_t = \Delta v + v$$

Then maximum principle applies. C^2-estimate follows from Yau's work on the Monge-Ampere equation. The C^3-estimate uses Calabi's computation. One can also use Evans' $C^{2,\alpha}$ argument.

b) Noncompact case:

Short time existence: W-X. Shi proved general existence theory for Ricci flow. Namely, the boundedness of curvature tensor for the initial metric implies short time existence.

Long time existence: W-X. Shi proved the following result.

Theorem (Shi) Let $(M, g(0))$ be a complete Kaehler manifold with nonnegative bisectional curvature. Assume that there exists a constant $C > 0$, and $\theta > 0$ such that for all $x \in M$

$$k(x,r) = \frac{1}{V(B_x(r))}\int_{B_x(r)} R(y)\,dv \le \frac{C}{(1+r)^\theta}. \tag{2}$$

Then **KRF** has long time existence.

Recently, Tam-N generalizes the result slightly to the case $k(x,r) = o(1)$. We also come up with a simple argument for the case $\theta > 1$ in Shi's theorem, which maybe enough for applications, say, the uniformization theorem for noncompact Kaehler manifolds with positive curvature.

Solving Poincaré-Lelong equation and **KRF**:

$$\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} u_0(x) = R_{\alpha\bar{\beta}}(x). \qquad \textbf{(PL)}$$

Tam-N: Under the assumption of Shi's theorem with $\theta > 0$, **PL** cab be solved with solution $u_0(x)$ satisfying:

$$|\nabla u_0|(x) \le C_1$$

for some $C_1 = C_1(C, m)$.

Simple calculation shows that $u(x,t) = u_0(x) - \log(\frac{\det(g_{\alpha\bar{\beta}}(x,t))}{\det(g_{\alpha\bar{\beta}}(x,0))})$ satisfies

$$\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} u(x,t) = R_{\alpha\bar{\beta}}(x,t).$$

Moreover $u(x,t)$ satisfies the time-dependent heat equation.

Then Bochner formula gives that

$$\left(\Delta - \frac{\partial}{\partial t}\right)\left(|\nabla u|^2 + R\right)(x,t) = \|u_{\alpha\beta}\|^2.$$

Simple consequence is that

$$\sup_{x\in M}\left(|\nabla u|^2 + R\right)(x,t) \leq \sup_{x\in M}\left(|\nabla u|^2 + R\right)(x,0) \qquad (*)$$

with equality holds if and only if $(M, g(t))$ is Kaehler-Ricci soliton. (*) implies the uniform curvature bound which implies the long time existence. The proof was motivated by an earlier work of Chow on gradient estimate on

Kaehler-Ricci flow. The argument works in compact case and give a proof of Cao's long time existence in case $biK \geq 0$ without using Monge-Ampere estimates. In this case, argument is simpler since one has potential for free.

3. Monotonicity-I

Monotonicity is used in general sense as Perelman's lecture. The very useful result is the following Li-Yau-Hamilton inequality:

Theorem (Cao). Let $(M, g(t))$ be KRF with nonnegative bisectional curvature. Then

$$\Delta R_{\alpha\bar{\beta}} + R_{\alpha\bar{\beta}\gamma\bar{\delta}} R_{\bar{\gamma}\delta} + \nabla_{\gamma} R_{\alpha\bar{\beta}} X_{\bar{\gamma}} + \nabla_{\bar{\gamma}} R_{\alpha\bar{\beta}} X_{\gamma}$$
$$+ R_{\alpha\bar{\beta}\gamma\bar{\delta}} X_{\bar{\gamma}} X_{\delta} + \frac{1}{t} R_{\alpha\bar{\beta}} \geq 0.$$

In case M is noncompact, we assumes that the curvature tensor is bounded.

Taking trace one does has the monotonicity of $tR(x,t)$.

The result was Kaehler version of the corresponding result of Hamilton on Ricci flow.

The underlying reason, in term of Chow-Chu's space-time formulation, for such estimate is the following result of Bando and Mok.

Theorem (Bando, Mok) Let $(M, g(t))$ be a KRF solution such that $g(0)$ has bounded nonnegative bisectional curvature. Then $(M, g(t))$ has nonnegative bisectional curvature.

Chow-Chu showed, for Riemannian case, that the LYH quantity is the curvature of some space time metrics on $M \times [0, T)$, which become zero at the initial time. If one can have similar construction of Chow-Chu for the Kaehler metric then Cao's theorem can be viewed as space-time version of the result of Bando-Mok.

4. Applications

The general goal is do uniformization and construct canonical metrics.

Theorem (Cao) Let M be a compact Kaehler manifold with $c_1(M) = 0$ ($c_1(M) = -[\omega]$). Then **KRF** converges to the Ricci flat metric (Kaehler-Einstein metric).

The existence results have been solved by Yau (Aubin-Yau) before. The **KRF** provides a flow proof. The method follow very closely the elliptic proof (via Monge-Ampere).

Mok made use of **KRF** in his celebrated result on the uniformization of compact non-negative bisectional curvature.

Theorem (Mok) Let M^m be a compact simply-connected Kähler manifold with non-negative bisectional curvature. Then M is biholomorphic to products of compact Hermitian symmetric spaces.

The positive case was proved by Siu-Yau, Mori earlier.

Noncompact complete case:

Theorem (Shi) Let M be a complete Kähler with positive sectional curvature and

$$k(x, r) \leq \frac{C}{(1+r)^{1+\epsilon}}.$$

Then M is biholomorphic to a strictly pseudo-convex domain in $\mathbf{C}^m$.

5. Monotonicity -II

a) Linear trace Li-Yau-Hamilton inequality.

Chow-Hamilton (Invent. Math. 1997) proved the linear trace Harnack inequality first for **RF** with nonnegative curvature operator. The main point is that the Li-Yau-Hamilton inequality holds for symmetric tensors satisfying the so-called Lichnerowizc heat equation. It reveals strong connections between the linear heat equation and the **RF**.

In the case $n = 2$ (m=1), it says that

$$\Delta \log u + R + \frac{1}{t} \geq 0$$

if $u(x,t)$ is a positive solution to time-dependent Shrödinger equation $(\frac{\partial}{\partial t} - \Delta - R)u(x,t) = 0$. Let $u = R$. One recovers Hamilton's LYH inequality for the surface.

Moreover it also contains Li-Yau's gradient estimate

$$\Delta \log u + \frac{1}{t} \geq 0.$$

The reason is that one can slow down of the Ricci flow by τ. Namely:

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2\tau R_{ij}(x,t).$$

Then for positive solution to $(\frac{\partial}{\partial t} - \Delta - \tau R)u(x,t) = 0$, Chow proves that

$$\Delta \log u + \tau R + \frac{1}{t} \geq 0.$$

Taking $\tau \to 0$ one recovers Li-Yau's inequality.

Since Li-Yau's estimate implies the Laplacian comparison theorem, one can think the Riemannian geometry (Kaehler geometry) is the limiting case of the **RF** (**KRF**) geometry.

Recently, we have been able to generalize Chow result to high dimension in Kaehler setting.

Theorem (N) Let $(M, g(t))$ be a solution to **KRF** with speed τ, with bounded non-negative bisectional curvature. Let $h_{\alpha\bar{\beta}}(x,t)$ be the symmetric tensor satisfying the Lichnerowicz heat equation. Then $Z^{(\tau)}(x,t) \geq 0$. Moreover, the equality holds for some $t > 0$ implies that $(M, g(t))$ is an expanding soliton if $h_{\alpha\bar{\beta}}(x,t) > 0$ and M is simply-connected.

Here $Z^{(\tau)}(x,t) = Z(x,t) + \tau \left(g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R_{\alpha\bar{\delta}} h_{\gamma\bar{\beta}} \right)(x,t)$. and

$$\begin{aligned} Z &= \frac{1}{2}\left(g^{\alpha\bar{\beta}} \nabla_{\bar{\beta}} div(h)_\alpha + g^{\gamma\bar{\delta}} \nabla_\gamma div(h)_{\bar{\delta}} \right) \\ &\quad + g^{\alpha\bar{\beta}} div(h)_\alpha V_{\bar{\beta}} + g^{\gamma\bar{\delta}} div(h)_{\bar{\delta}} V_\gamma \\ &\quad + g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} h_{\alpha\bar{\delta}} V_{\bar{\beta}} V_\gamma + \frac{H}{t}. \end{aligned}$$

$\tau = 0$ **case is a new inequality, which implies the differential Harnack for Hermitian-Einstein flow, and has been proven to be useful.**

($\tau = 1$ case, was proved before by Tam-N, gives Cao's trace differential Harnack. in-equality if $h_{\alpha\bar\beta} = R_{\alpha\bar\beta}$).

b) Integral quantities-mostly compact case:

For **NKRF** and $c_1(M) = [\omega]$ case:

i) Mabuchi's K-energy:

$$\begin{aligned}\nu_\omega(\phi) &= \int_M \log\left(\frac{\omega_\phi^m}{\omega^m}\right) + \int_M f_\omega(\omega^m - \omega_\phi^m) \\ &- \sum_{i=0}^{m-1} \int_M \left(\frac{m-i}{m+1}\right) \sqrt{-1}\partial\phi \wedge \bar\partial\phi \wedge \omega^i \wedge \omega_\phi^{m-i-1}.\end{aligned}$$

Here we normalize the volume to be 1, f_ω is the normalized potential function. $\nu_\omega(\phi)$ is monotone decreasing along the flow.

ii) Ding-Tian's F-functional:

$$F_\omega(\phi) = \nu_\omega(\phi) + \int_M f_{\omega_\phi}\omega_\phi^m - \int_M f_\omega \omega^m.$$

iii) Chen-Tian's functionals for the case $biKm \geq 0$.

For general **RF** on compact manifolds:

Perelman's energy and entropy:

$$\lambda(t) = \inf_{\int_M v^2\,dv=1} \int_M (4|\nabla v|^2 + Rv^2)\,dv$$

and

$$\begin{aligned}\mu(\tau) \quad = \quad & \inf_{\int_M v^2\,dv=1} \int_M \Big[\tau\left(4|\nabla v|^2 + R\right) \\ & - \log(v^2)v^2 - m\log(\pi\tau)\Big]\,dv\end{aligned}$$

with $\tau = T_0 - t$. Both $\lambda(t)$ and $\mu(\tau)$ are isoperemetric constants. They are monotone increasing in t.

6. Large-time behavior of the KRF

Compact case: Nothing is known in general case. No much known about the singularity. Hard to do the surgery in the Kaehler case.

Special case $c_1(M) = [\omega]$: It was claimed by Perelman that if there exists a KE metric in the class $[\omega]$ then **NKRF** converges to a KE metric.

Chen-Tian: The case M has positive bisectional curvature. Again assuming $[\omega]$ contains a KE metric.

Noncompact case: Not much in general.

Special case: M has bounded nonnegative bisectional curvature. The **KRF** become degenerate as $t \to \infty$. What normalization will ensure convergence to a flat metric? Chau–Tam proves convergence, using Shi's idea, under some assumptions on the $g_{\alpha\bar{\beta}}(x_0, t)$ for all t.

7. Applications of Perelman's entropy formula

Theorem (Perelman) For **NKRF**, assume that $R(x, 0) > 0$. Then there exists a constant $C_1(g(0), m)$ such that

$$R(x, t) \leq C_1$$

and

$$D(t) := \text{Diameter}(M, g(t)) \leq C_1.$$

The result serves an important step towards the convergence result claimed by Perelman.

The special case, when M has nonnegative bisectional curvature, of the above result was proved by Cao-Chen-Zhu, via Perelman's non-collapsing theorem.