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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SRHERMES @ BRANDEIS.EDU

Speaker's Name: R. Marsh

Talk Title: Reflection Group Presentations Arising from Cluster Algebras.

Date: 10 / 30 / 12 **Time:** 9 : 30 (am) / pm (circle one)

List 6-12 key words for the talk: Reflection Groups, cluster algebras, Root systems, Dynkin diagrams, coxeter groups

Please summarize the lecture in 5 or fewer sentances: The speaker introduced reflection groups in terms of an arbitrary seed in a cluster algebra. If the seed has a Dynkin quiver the usual Weyl group is obtained. The speaker introduced the notion of companion bases and showed that these groups admit companion bases in finite type.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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Reflection Group Presentations Arising from Cluster Algebras

R. Marsh October 30, 2012

Joint with M. Barot.

Main Idea: We want to define a finite crystallographic reflection group via a presentation using *any* seed in the corresponding cluster algebra. In the case that the seed has a Dynkin quiver, the presentation corresponding to this seed is the usual Coxeter group presentation.

Motivation:

1. (Barot-Geiß-Zelevinsky) Criteria for a cluster algebra to be of finite type in terms of an arbitrary seed.
2. (M. Parsons) Companion bases in the root system (bases whose inner product give the quiver in the seed). Gives a partial generalisation of Gabriel's Theorem for cluster-tilted algebras (in type A) (c.f. also Ringel).
3. (Cameron-Seidel-Tsaranov) Give presentations of reflection groups (in the simply-laced case) coming from $\mathbb{Z}$-bases of the $\mathbb{Z}$-span of roots.

1 Cluster Algebras

Cluster algebras are a class of commutative algebras defined by Fomin-Zelevinsky in 2001. Introduced to model the duel canonical basis of a quantum group e.g. $\mathbb{C}_q[\mathfrak{sl}_n]$ and total positivity.

Let B be an $n \times n$ skew-symmetrisable integer matrix, i.e., there is a diagonal matrix D with positive integer diagonal entries so that DB is skew-symmetric. This defines a cluster algebra $A(B)$. (In particular, no frozen variables.)

1.1 Mutation of B

Fix a $k \in \{1, \dots, n\}$. Define a new matrix $B' = \mu_k(B)$ where

$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k \\ B_{ij} + \frac{|B_{ik}|B_{kj} + B_{ik}|B_{kj}|}{2} & \text{else} \end{cases}$$

Definition 1.1. The *diagram* $\Gamma(B)$ of B is the edge-weighted quiver with vertices $1, \dots, n$. There is an arrow $i \to j$ iff $B_{ij} > 0$ and it is weighted $|B_{ij}||B_{ji}|$.

A Dynkin diagram Δ gives rise to an unoriented diagram $\widetilde{\Delta}$ with the same vertices by ignoring the orientation and replacing multiple edges with edges of the same weight.

Example 1.2. The B_2 diagram becomes $\circ \overset{2}{\text{———}} \circ$.

Theorem 1.3 (FZ2)**.** The cluster algebra $A(B)$ is of finite type if and only if there is a matrix B' mutation equivalent to B such that $\Gamma(B') = \widetilde{\Delta}$ with some orientation.

This gives a bijection between cluster algebras of finite type (up to strong isomorphism) and Dynkin diagrams.

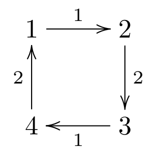
Mutation of matrices induces mutation of diagrams. If we focus on the finite type case, then the mutation is given by:

1. Reverse orientations on all edges incident to k.

2. If there is a path $i \xrightarrow{a} k \xrightarrow{b} j \xrightarrow{c} i$ (allow $c = 0$ if there is no edge) then mutate to $i \xleftarrow{a} k \xleftarrow{b} j \xleftarrow{c'} i$ where $c + c' = \max\{a, b\}$.

Example 1.4. Given $Q = 1 \xrightarrow{1} 2 \xrightarrow{1} 3$ then $\mu_2 Q = 1 \xleftarrow[1]{} 2 \xleftarrow[1]{} 3$, with an arrow $1 \xrightarrow{1} 3$.

The quiver



is mutation equivalent to F_4.

2 Main Result

Let Γ be a diagram coming from a cluster algebra of finite type. Define

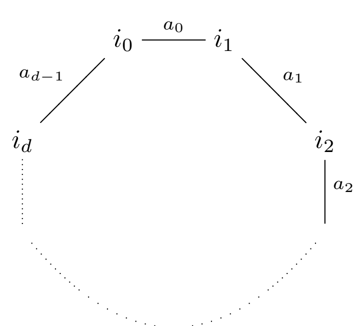
$$m_{ij} = \begin{cases} 2 & \text{if } i, j \text{ not connected} \\ 3 & \text{if } i \overset{1}{\text{———}} j \\ 4 & \text{if } i \overset{2}{\text{———}} j \\ 6 & \text{if } i \overset{3}{\text{———}} j \end{cases}$$

Let W_Γ be the group with generators $s_1, \ldots, s_n$ subject to the relations

1. $s_i^2 = e$ for each i

2. $(s_i s_j)^{m_{ij}} = e$ for each $i \neq j$.

3. For any chordless cycle in the underlying unoriented graph



where either $a_{d-1} = 2$ or $a_i = 1$ for every i, there is a relation

$$(s_{i_0} s_{i_1} \cdots s_{i_{d-1}} s_{i_{d-2}} \cdots s_{i_1})^2 = e.$$

Theorem 2.1 (BM)**.** Suppose Γ is mutation equivalent to an orientation of $\widetilde{\Delta}$ with Δ Dynkin. Then $W_\Gamma \cong W_\Delta$.

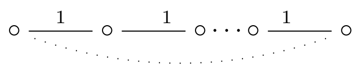
Remarks 2.2. 1. In the simply-laced case, using [BGZ, Parsons] one can see these presentations were found in [CST].

2. For a given chordless cycle, any one relation of the third type implies all the others for that cycle using the first two relations only. In particular, this can be done for the ones in the other direction.

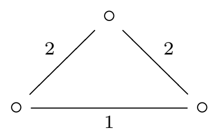
3. Any chordless cycle is oriented [FZ].

4. If the weight condition in the third relation fails, the relation we get is $s_{i_0}s_{i_1}\cdots s_{i_{d-1}}s_{i_{d-2}}\cdots s_{i_1})^3 = e$ (follows from the other three).

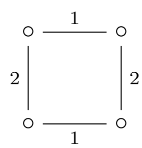
5. The only possible chordless cycles (ignoring orientation) are [FZ]:
Type D_n:



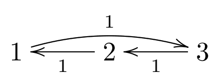
Type B_3:



Type F_4:



6. If Γ is an orientation of $\widetilde{\Delta}$ then there are no cycles, and so we get the usual Coxeter presentation.

Example 2.3. If $\Gamma =$  we get

$$W_\Gamma = \langle s_1, s_2, s_3 : s_i^2 = e, (s_1s_2)^3 = (s_2s_3)^3 = (s_1s_3)^3 = (s_1s_2s_3s_2)^2 = e\rangle.$$

It turns out that $s_1s_2s_3s_2 = (12)(34), s_1 = (12), s_2 = (23), s_3 = (24)$ gives an example of an isomorphism with S_4.

3 Companion Bases

Definition 3.1. With M. Parsons, using work of Barot-Geiß-Zelevinksy. Suppose that B is an exchange matrix of finite type and Φ the corresponding root system. A $\mathbb{Z}$-basis $\{\beta_1, \ldots \beta_n\} \subset \Phi$ for $\mathbb{Z}\Phi$ is called a *companion basis* for B if $|(\beta_i, \check{\beta}_j)| = |B_{ij}|$ for $i \neq j$.

Can get dimension vectors for indecomposable modules over cluster-tilted algebras (Parsions did Type A and D_4). c.f. also, Ringel.

Theorem 3.2 (Parsions, using BGZ)**.** In the finite type case, a companion basis always exists. In the presentation above, $s_i \mapsto s_{\beta_i}$ under the isomorphism $W_\Gamma \to W_\Delta$ where $\{\beta_1, \ldots, \beta_n\}$ is a companion basis for B with $\Gamma(B) = \Gamma$.

In the simply-laced case, $\beta_1, \ldots, \beta_n$ gives a *signed graph* structure on Γ. That is, a function $f : \Gamma_1 \to \{\pm\}$ (Γ_1 is the edges of Γ) by labeling an edge $i \to j$ with the sign of (β_i, β_j). Then (Γ, f) has an odd number of +'s in each chordless cycle ([BGZ],[CST]).

Suppose $\beta_1, \ldots, \beta_n$ is a companion basis for B and (Γ, f) the corresponding signed diagram. Fix k and take

$$\beta_i' = \begin{cases} s_{\beta_k}(\beta_i) & i \to k \\ \beta_i & \text{else} \end{cases}$$

Then by Parsons, BGZ, BM $\beta_1', \ldots, \beta_n'$ is a companion basis for $\mu_k B = B'$. This gives a new signed diagram (Γ', f').

Proposition 3.3 (BM)**.** (Γ', f') is obtained from (Γ, f) by *local switching* at the vertex k. Here we are considering (Γ, f) and (Γ', f') as a signed graph (no orientation).