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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SRHERMES@BRANDEIS.EDU

Speaker's Name: T. Nakanishi

Talk Title: Diagrammatic Description of c-Vectors and d-Vectors of Cluster Algebras of Finite Type

Date: 10 / 30 / 12 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Cluster Algebras, c-Vectors, d-Vectors, Finite Type, Dynkin Diagrams

Please summarize the lecture in 5 or fewer sentances: The Speaker introduced d-vectors and positive c-vectors. He showed that they can be explicitly described by diagrams in finite type and gave a complete list of these diagrams. He explained several consequences of this work.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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- **Computer Presentations**: Obtain a copy of their presentation
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Diagrammatic Description of c-Vectosr and d-Vectors of Cluster Algebras of Finite Type

T. Nakanishi October 30, 2012

Joint with Salvatore Stella (arXiv:1210.6299)

1 c- and d-Vectors

Let B be a skew-symmetriseable matrix. Can associate to B a cluster algebra $\mathcal{A}_\bullet(B)$ with principal coefficients.

Let $x = (x_1, \dots, x_n)$ be an initial cluster. Then any cluster variable x' can be written in the form

$$x' = \frac{p(x)}{x_1^{d_1} \cdots x_n^{d_n}}$$

where $p(x)$ is a polynomial in the cluster variables of x not divisible by any of the x_i. The vector $d = (d_1, \cdots, d_n)$ is the *d-vector* of x'.

Consider the matrix $\widetilde{B} = \begin{pmatrix} B \\ I \end{pmatrix}$. When mutated we get a matrix of the form $\widetilde{B}' = \begin{pmatrix} B' \\ C' \end{pmatrix}$. The j-th column of C' is the j-th *c-vector*.

There is an alternate definition via tropicalization. Let $\pi_x : \mathbb{Q}_+(x) \to \mathrm{Trop}(x)$ be the semifield homomorphism given by $x \mapsto x$ ($\mathrm{Trop}(x)$ the tropical semifield). Under this homomorphism

$$\pi_x(x'_j) = \prod_{i=1}^{n} x_i^{-d_{ij}}$$

where $(d_{ij})_{i=1}^n$ is the j-th d-vector and

$$\pi_y(y'_j) = \prod_{i=1}^{n} y_i^{c_{ij}}$$

where $(c_{ij})_{i=1}^n$ is the j-th c-vector.

Conjecture 1.1 (Sign-Coherence Conjecture (Fomin-Zelevinsky '07))**.** Any non-initial d-vector is a positive vector. Any c-vector is either a positive or a negative vector (i.e. there do not occur mixed signs).

The second part has been recently proven for the skew-symmetric case.

There is an identification:

$$\{\text{cluster algebras}\} \leftrightarrow \{\text{Kac-Moody algebras}\}$$

Given by $B \mapsto A(B)$ (the Cartan counterpart of B, e.g. $\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$) and taking the corresponding root system $\Delta(A(B))$.

Conjecture 1.2 (NS and Zelevinsky)**.** Any c-vector is a root of $\Delta(A(B))$.

Theorem 1.3 (Náejera Chávez '12)**.** This is true for B skew-symmetric.

For d-vectors, this is not true (counter examples by Marsh-Reiten; to appear).

2 Finite Type

Below we assume $\mathcal{A}_\bullet(B)$ is of finite type. It is well-known that these are classified by the Dynkin diagrams X of type $A_n, \ldots, G_2$ (iff B is mutation equivalent to B' so that the Cartan counterpart $A(B')$ is of type X). If B satisfies this condition, we say B is type X.

Example 2.1. $\xymatrix{1 \ar[r] & 2 \ar[r] & 3 \ar[r] \ar@/_1pc/[ll] & 4}$ is mutation equivalent to $1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4$ and so is of finite type.

Let $\mathcal{D}(B)$ be the set of non-initial d-vectors of $\mathcal{A}_\bullet(B)$ and $\mathcal{C}_+(B)$ the set of positive c-vectors of $\mathcal{A}_\bullet(B)$.

Theorem 2.2 (Fomin-Zelevinsky (FZ2))**.** Suppose $A(B)$ is of type X, and B is bipartite (i.e. every vertex is a source or a sink). Then $\mathcal{D}(B) = \Delta_+(A(B))$ (the set of positive roots of the corresponding root system).

Theorem 2.3 (Kac '80)**.** For B skew-symmetric, let $Q(B)$ be the corresponding quiver, and k an algebraically closed field. Then α is the dimension vector of some indecomposable kQ-module if and only if α is a positive root of $A(B)$.

Recall a cluster tilted algebra $\Lambda(B)$ is a quotient of $kQ(B)$ by some explicit relations.

Theorem 2.4. For B of type A, D, E

1. (BMR '07, Caldero-Chapoton-Schiffler '06) $\mathcal{D}(B)$ is the set of dimension vectors of indecomposable modules over $\Lambda(B)$ (denote this set by $\mathcal{D}im(B)$).
2. (Nájera Chávez '12) $\mathcal{C}_+(B) = \mathcal{D}im(B)$.

These two facts imply that $\mathcal{C}_+(B) = \mathcal{D}(B)$.

3 Main Result

Theorem 3.1 (NS)**.** For any B of cluster finite type we give a complete list of $\mathcal{D}(B)$ and $\mathcal{C}_+(B)$ by (certain generalized) Dynkin diagrams.

Consequences:

1. Get a list of $\mathcal{D}im(B)$ for A, D, E.
2. Root property for B, C, F, G.
3. $\mathcal{D}(B) = \mathcal{C}_+(B)$ for B, C, F, G.
4. $|\mathcal{D}(B)|$ only depends on the cluster type.
5. $\mathcal{D}(B)$ only depends on $A(B)$, etc.