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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes Email/Phone: SRHERMES@BRANDEIS.EDU

Speaker's Name: M. Gekhtman

Talk Title: Cremmer-Gervais Cluster Algebras

Date: 11 / 2 / 12 Time: 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Cluster Algebras, Poisson-Lie algebras, Belavin-Drinfel'd data, Algebraic groups, Yang-Baxter Equation, Poisson Geometry

Please summarize the lecture in 5 or fewer sentances: The speaker introduced the notion of a compatible Poisson structure on a cluster algebra. He introduced Belavin-Drinfel'd data to a Poisson-Lie bracket and related this to the cluster algebra structure on the coordinate ring of a Lie group.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

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Cremmer-Gervais Cluster Algebras

M. Gekhtman November 2, 2012

Joint with M. Shaprio, and A. Vainshtein.

Motivation:

1. Want to describe non-isomorphic cluster structures that are supported in a coordinate ring of a given variety.

2. Investigate "Poisson-Lie" features of cluster algebras.

Main Tool: Compatible Poisson structure in cluster algebras: Let $(\mathbf{x}, B)$ be an initial seed of some cluster algebra of geometric type, where $\mathbf{x} = \{\underbrace{x_1, \dots, x_m}_{\text{cluster}}, \underbrace{x_{m+1} \dots, x_n}_{\text{frozen}}\}$. Define a Poisson bracket $\{\ ,\ \}$ on the rational functions in $x_1, \dots, x_n$ by $\{x_i, x_j\} = \omega_{ij} x_i x_j$ for some $\omega_{ij} \in \mathbb{Z}$. The matrix $\Omega = (\omega_{ij})$ is skew-symmetric. (Many names for this Poisson structure: log-canonical, diagonal quadratic, etc.) We require that in any cluster $(\mathbf{x}', B')$ that $\{x'_i, x'_j\} = \omega'_{ij} x'_i x'_j$. Such a Poisson bracket is called *compatible* with $\mathcal{A}(B)$.

Condition for Compatibility: If B is non-degenerate, then compatibility is equivalent to the condition

$$\Omega B = \begin{pmatrix} D \\ 0 \end{pmatrix}$$

where D is the skew-symmetriser.

Remark 0.1. Ω is not unique, but given one such Ω one can describe all others using the *global toric action* on $\mathcal{A}(B)$. The torus $(\mathbb{C}^*)^{n-m}$ acts on $\mathbf{x}$ by

$$x_i \mapsto x_i \prod_{s=1}^{n-m} t_s^{\omega_{m+s,i}}.$$

Strategy: Given a Poisson variety $(V, \{\ ,\ \})$

1. Find log-canonical coordinate system made of regular functions.

2. Construct the matrix B.

3. Check that $\mathcal{O}(V)$ contains $\mathcal{A}_{\mathbb{C}}(B)$.

4. Show that $\mathcal{O}(V)$ is contained in either$\mathcal{A}_{\mathbb{C}}(B)$ or the upper cluster algebra $\overline{\mathcal{A}}_{\mathbb{C}}(B)$.

1 Poisson-Lie Groups and Belavin-Drinfel'd Classification

Definition 1.1. A Lie group G equipped with a Poisson bracket $\{\ ,\ \}$ is a *Poisson-Lie group* if the multiplication map $(x, y) \mapsto xy$ is Poisson. (Studied by Sklyanin, Drinfel'd.)

Example 1.2. $B_2^+ = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : \{a, b\} = ab \right\}$ is a Poisson-Lie group.

From now on, assume G is a simple Lie group, $\mathfrak{g}$ its Lie algebra, and $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ its simple positive roots. We deal with factorizable quasi-triangular Poisson-Lie groups (which can be described through the Belavin-Drinfel'd classification).

The structure constants of the Poisson-Lie bracket $\{\ ,\ \}$ can be "packed" into $r \in \mathfrak{g} \otimes \mathfrak{g}$ (equivalently $R \in \mathrm{End}(\mathfrak{g})$). The element r must satisfy the *classical Yang-Baxter equation.* Belavin-Drinfel'd showed how to construct r using *Belavin-Drinfel'd data*:

$$(\Gamma_1 \xrightarrow{\gamma} \Gamma_2, r_0)$$

where $\Gamma_1, \Gamma_2 \subset \Pi$, γ is an isometry satisfying the nilpotency condition $\forall \alpha \in \Gamma_1,\ \exists m > 0$ s.t. $\gamma^m(\alpha) \notin \Gamma_1$, and $r_0 \in \mathfrak{h} \wedge \mathfrak{h}$ (Cartan) satisfies some linear equation determined by $\gamma : \Gamma_1 \to \Gamma_2$.

Example 1.3. The standard Poisson-Lie structure corresponds to $\Gamma_1 = \Gamma_2 = \varnothing$ and r_0 is arbitrary.

Indication:

1. Regular Poisson submanifolds of G are double Bruhat cells (Rogan-Zelevinsky).
2. Standard $\{\ ,\ \}$ is compatible with cluster structure defined by [BFZ] on double Bruhat cells.

Theorem 1.4 (GSV)**.** There is a cluster structure on $\mathcal{O}(G)$ such that:

1. the number of frozen variables is 2ℓ and the exchange matrix is non-degenerate (i.e. of full rank),
2. the upper cluster algebra $\overline{\mathcal{A}}(B) = \mathcal{O}(G)$,
3. there is a global toric action of $(\mathbb{C}^*)^{2\ell}$ on $\mathcal{A}(B)$ induced by the natural action of $H \times H$ (Cartan) on G,
4. any Poisson-Lie bracket in a trivial B.-D. class is compatible,
5. any Poisson-Lie bracket compatible with $\mathcal{A}(B)$ is in the trivial B.-D. class.

Conjecture 1.5 (GSV)**.** There is a cluster structure on $\mathcal{O}(G)$ such that:

1. the number of frozen variables is $2(\ell - |\Gamma_1|)$ and the exchange matrix is non-degenerate (i.e. of full rank),
2. the upper cluster algebra $\overline{\mathcal{A}}(B) = \mathcal{O}(G)$,
3. there is a global toric action of $(\mathbb{C}^*)^{2(\ell - |\Gamma_1|)}$ on $\mathcal{A}(B)$ induced by the natural action of $H_\gamma \times H_\gamma$ (Cartan) on G,
4. any Poisson-Lie bracket in a trivial B.-D. class is compatible,
5. any Poisson-Lie bracket compatible with $\mathcal{A}(B)$ is in *this* trivial B.-D. class.

Initial Evidence: True for SL_3 and SL_4.

2 Cremmer-Gervais Cluster Algebras and Exotic Cluster Structure in SL_n and GL_n

C.-G.-B.-D. Data (for $G = SL_n$): $\gamma : \Gamma_1 = \{\alpha_2, \dots, \alpha_{n-1}\} \to \Gamma_2 = \{\alpha_1, \dots, \alpha_{n-2}\}$

Typically gives more complicated Poisson brackets than the standard B.-D. data.

Strategy:

1. Initial Cluster: Drinfel'd double $D = G \times G$. r goes to maps $r_{\pm} : G \to G_{\pm}$ (subgroups of G) which gives (almost) factorization $D = G_r \times d(G)$ where

$$G_r = \{(r_+(x), r_-(x)) : x \in G\} \qquad \text{and} \qquad d(G) = \{(x, x) : x \in G\}.$$

   Regular Poisson submanifolds: intersection of right/left orbits of G_r in D with $d(G)$.

2. Induction $GL_n \to GL_{n-1}$

3. Poisson anti-involution: $X \to w_0 X w_0$