NOTETAKER CHECKLIST FORM
(Complete one for each talk.)

Name: Elizabeth Gross  Email/Phone: egross7@uic.edu

Speaker's Name: Uwe Nagel

Talk Title: Enumerations deciding the Weak Lefschetz Property

Date: 12/4/2012  Time: 11:30 am / pm (circle one)

List 6-12 key words for the talk: weak lefschetz property, monomial
ideals, lozenge tilings, lattice paths,Mahonian determinants,
syzygy bundles, Laplace equations, Togliatti system
Please summarize the lecture in 5 or fewer sentences:
- Discusses an approach for studying monomial ideals
  in three variables using lozenge tilings.
- Gives a combinatorial interpretation of
  the weak lefschetz property. Explores Laplace
  equations as an application.

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(YYYY.MM.DD.TIME.SpeakerLastName)

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line.
Enumerations of the Weak Lefschetz Property

joint work with
David Cook II
(University of Notre Dame)

Uwe Nagel
(University of Kentucky)

MSRI, December 4, 2012
Outline

- Lefschetz Properties
- Lozenge tilings, perfect matchings, and lattice paths
- Mahonian Determinants
- Type 2 algebras
- Existence of Laplace equations
Lefschetz Properties

\[ R = K[x_1, \ldots, x_n], \, K \text{ an infinite field} \]
\[ I \subset R \text{ homogeneous, artinian ideal (dim}_K R/I < \infty) \]

Definition

A \[ = R/I \] has the Weak Lefschetz Property (WLP) if there is a linear form \( \ell \in R \) such that the multiplication \( \times \ell: [A] \rightarrow [A]_{i+1} \) has maximal rank for all \( i \) (i.e. is injective or surjective).

A has the Strong Lefschetz Property (SLP) if \( \times \ell^d: [A] \rightarrow [A]_{i+d} \) has maximal rank for all \( i \) and \( d \).

Remark:
(i) \( \ell \) general.
(ii) WLP implies restrictions on Hilbert function (\( g \)-Theorem (Stanley)).
(iii) WLP and SLP are related to Fröberg’s conjecture.
Lefschetz Properties

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\( (g\text{-Theorem (Stanley))} \).

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Known results

**Theorem**

- (Harima, Migliore, N., Watanabe, 2003) If $n \leq 2$ and $\text{char } K = 0$, then $A$ has the SLP.
- (Migliore, Zanello, 2007) If $n \leq 2$, then $A$ always has the WLP.
Known results

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Theorem (Stanley, 1980; ...)

If $\text{char } K = 0$, then each monomial c.i., $I = (x_1^{a_1}, \ldots, x_n^{a_n})$, has the SLP.
<table>
<thead>
<tr>
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Monomial ideals in three variables

\( I \subset R = K[x, y, z] \) artinian monomial ideal.
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**Theorem (Boij, Migliore, Miró-Roig, N., Zanello, 2012)**

*If $n = 3$, char $K = 0$, and $R/I$ is level of type 2, then $R/I$ has the WLP.*

Counterexamples if $R/I$ is not level or if char $K > 0$. 
Monomial ideals in three variables

$I \subset R = K[x, y, z]$ artinian monomial ideal.

**Theorem (Boij, Migliore, Miró-Roig, N., Zanello, 2012)**

If $n = 3$, char $K = 0$, and $R/I$ is level of type 2, then $R/I$ has the WLP.

Counterexamples if $R/I$ is not level or if char $K > 0$.

**Example**

- If $I = (x^7, y^7, z^7, x^2y^2z^2)$, then $R/I$ has the WLP if and only if the characteristic of $K$ is not 2 or 7.
Monomial ideals in three variables

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- If $I = (x^7, y^7, z^7, x^2y^2z^2)$, then $R/I$ has the WLP if and only if the characteristic of $K$ is not 2 or 7.
- If $I = (x^{20}, y^{20}, z^{20}, x^3y^8z^{13})$, then $R/I$ has the WLP if and only if the characteristic of $K$ is not 2, 3, 5, 7, 11, 17, 19, 23, or
Monomial ideals in three variables

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Triangular region $\mathcal{T}_d$: equilateral triangle of side length $d$, subdivided into equilateral unit triangles:

- $\binom{d}{2}$ downward-pointing (▽) - labeled by monom. in $[R]_{d-2}$, and
- $\binom{d+1}{2}$ upward-pointing (△) - labeled by monom. in $[R]_{d-1}$. 

**Diagrams:**

- $\mathcal{T}_2$: $x$, $y$, $z$
- $\mathcal{T}_3$: $x^2$, $xy$, $xz$, $y$, $y^2$, $yz$, $z$, $z^2$
- $\mathcal{T}_4$: $x^3$, $x^2y$, $x^2z$, $xy^2$, $xy$, $xyz$, $xz$, $xz^2$, $y^2z$, $yz^2$, $z^3$
$I \subset R$ any monomial ideal
$d \geq 1$ any integer
triangular region $T_d(I)$: obtained from $\mathcal{T}_d$ by removing triangles with labels in $I$.

Example 1

$I = (xy, y^2, z^3)$, $d = 4$. 

$\mathcal{T}_4$  

$T_4(xy, y^2, z^3)$
Example 2

\[ I = (x^a y^b z^c). \]
Lozenge tilings

\( T \subset T_d \) any subregion

Lozenge (diamond, callisson, rhombus):
glue an \( \nabla \)- and an \( \triangle \)-triangle along the common edge
Lozenge tilings

\( T \subset T_d \) any subregion

**Lozenge** (diamond, callisson, rhombus):

Glue an \( \bigtriangledown \)- and an \( \bigtriangleup \)-triangle along the common edge

Tile \( T \) by lozenges if possible

A tiling of \( T_8(x^7, y^7, z^6, xy^4z^2, x^3yz^2, x^4yz) \)

Necessary tileability condition: balanced (\( \#\bigtriangledown = \#\bigtriangleup \))
$T \subset T_d$ any subregion

$G(T)$ bipartite graph:
- $B =$ set of centers of $\triangledown$-triangles, ordered revlex by labels,
- $W =$ set of centers of $\triangle$-triangles, ordered revlex by labels
- Vertices: $B \cup W$
- Edges: $(B_i, W_j)$ if the corresponding triangles share an edge

Bi-adjacency matrix $Z(T)$: zero-one matrix of size $\#B \times \#W$:

$$Z(T)_{(i,j)} = \begin{cases} 
1 & \text{if } (B_i, W_j) \text{ is an edge} \\
0 & \text{otherwise}
\end{cases}$$
Assume $T$ is balanced ($\#B = \#W$):

**Perfect matching** of $G(T)$: a set of pairwise non-adjacent edges of $G(T)$ such that each vertex is matched
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\[
\begin{array}{c}
1 - 1 \\
\end{array}
\]

lozenge tiling of $T$
Proposition

If $T$ is balanced, then

$$
\text{#lozenge tilings of } T = \text{#perfect matchings} = \text{perm } Z(T).
$$
Proposition

If $T$ is balanced, then

$$\#\text{lozenge tilings of } T = \#\text{perfect matchings} = \text{perm } Z(T).$$

Definition

A lozenge tiling $\tau$ of $T$ induces a bijection $B \to W$, $B_i \mapsto W_{\sigma(i)}$, where $\sigma \in S_{\#B}$. The perfect matching sign of $\tau$ is

$$\text{msgn } \tau := \text{sgn } \sigma.$$
Proposition

If \( T \) is balanced, then
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Definition

A lozenge tiling \( \tau \) of \( T \) induces a bijection \( B \to W, B_i \mapsto W_{\sigma(i)} \), where \( \sigma \in \mathfrak{S}_{\#B} \). The perfect matching sign of \( \tau \) is
\[
\text{msgn } \tau := \text{sgn } \sigma.
\]

Corollary

\[
\sum_{\tau \text{ tiling of } T} \text{msgn } \tau := \text{det } Z(T).
\]
Example

Consider $T = T_6(x^3, y^4, z^5)$.

$Z(T) = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$.

perm $Z(T) = \det Z(T) = 10$. 
$T \subset T_d$ any subregion

$L(T)$: set of midpoints of NE edges of triangles in $T$

- Label the vertices of $L(T)$ that are only on $\triangle$-triangles by $A_1, \ldots, A_m$ according to the revlex order of the monomials, beginning with the smallest.
- Label the vertices of $L(T)$ that are only on $\triangle$-triangles by $E_1, \ldots, E_n$ according to the revlex order of the monomials, beginning with the smallest.

A **lattice path** from $A_i$ to $E_j$ is a path in $L(T)$ where each single move is to the East ($\rightarrow$) or to the South-East ($\downarrow\rightarrow$).
**Lattice paths**

$T \subset T_d$ any subregion

$L(T)$: set of midpoints of NE edges of triangles in $T$

- Label the vertices of $L(T)$ that are only on $\triangle$-triangles by $A_1, \ldots, A_m$ according to the revlex order of the monomials, beginning with the smallest.

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Lattice path matrix $N(T)$: size $m \times n$

$$N(T)_{(i,j)} = \#\text{lattice paths in } \mathbb{Z}^2 \text{ from } A_i \text{ to } E_j.$$
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Assume $T$ is balanced ($m = n$):
family of non-intersecting lattice paths in $L(T)$ (from $A_1, \ldots, A_m$ to $E_1, \ldots, E_m$)
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Lattice paths

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**Definition**

The **lattice path sign** of a lozenge tiling $\tau$ of $T$ is

$$\text{lpsgn } \tau := \text{ sgn } \sigma,$$

where $\sigma \in \mathcal{S}_m$ is the permutation such that, for all $i$, the path starting in $A_i$ ends in $E_{\sigma(i)}$. 

**Theorem (Lindström, Gessel & Viennot)**

If $T$ is balanced, then

$$\sum_{\tau \text{ tiling of } T} \text{lpsgn } \tau = \det N(T).$$
Lattice path matrix $N(T)$: size $m \times n$

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If $T$ is balanced, then

$$\sum_{\tau \text{ tiling of } T} \text{lpsgn } \tau := \det N(T).$$
Example

\[ T = T_6(x^3, y^4, z^5) \text{ and its rotations.} \]
$T = T_d(I) \subset T_d$

$\tau$ lozenge tiling of $T$:

- perfect matching sign $\text{msgn}_\tau$ - enumerated by $\det Z(T)$
- lattice path sign $\text{lpsgn}_\tau$ - enumerated by $\det N(T)$
Comparisons

\[ T = T_d(l) \subset T_d \]

\( \tau \) lozenge tiling of \( T \):
- perfect matching sign \( \text{msgn}\, \tau \) - enumerated by \( \det Z(T) \)
- lattice path sign \( \text{lpsgn}\, \tau \) - enumerated by \( \det N(T) \)

**Theorem**

(a) Let \( \tau \) and \( \tau' \) be two lozenge tilings of \( T \). Then

\[ \text{msgn}(\tau) \cdot \text{lpsgn}(\tau) = \text{msgn}(\tau') \cdot \text{lpsgn}(\tau'). \]

(b)

\[ |\det Z(T)| = |\det N(T)|. \]
Comparisons

\( T = T_d(I) \subset T_d \)

\( \tau \) lozenge tiling of \( T \):

- perfect matching sign \( \text{msgn} \tau \) - enumerated by \( \det Z(T) \)
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\]

(b)

\[
| \det Z(T) | = | \det N(T) |.
\]

**Corollary**

If \( T \) is tileable and simply connected, then

\[
| \det Z(T) | = \text{perm } Z(T) > 0.
\]
Example

\[ T = T_6(x^3, y^4, z^5). \]

Then

\[ 10 = |\det N(T)| = |\det Z(T)| = \text{perm}(T). \]
Mahonian determinants

A $2 \times 6 \times 3$ plane partition. The associated lozenge tiling.

**Theorem (MacMahon)**

The number of plane partitions in an $a \times b \times c$ box is

$$
\text{Mac}(a, b, c) := \frac{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(a + b + c)}{\mathcal{H}(a + b)\mathcal{H}(a + c)\mathcal{H}(b + c)},
$$

where $\mathcal{H}(n) := \prod_{i=0}^{n-1} i!$ is the hyperfactorial of $n$. 

\begin{align*}
3 &\quad 3 &\quad 2 &\quad 2 &\quad 2 &\quad 1 \\
3 &\quad 2 &\quad 2 &\quad 1 &\quad 0 &\quad 0
\end{align*}
Proposition

If $T = T_d(x^a, y^b, z^c)$ is balanced, that is, $d = \frac{1}{2}(a + b + c)$ is an integer, then

$$|\det Z(T)| = \perm Z(T) = \text{Mac}(d - a, d - b, d - c).$$
Proposition

If \( T = T_d(x^a, y^b, z^c) \) is balanced, that is, \( d = \frac{1}{2}(a + b + c) \) is an integer, then

\[ |\det Z(T)| = \text{perm } Z(T) = \text{Mac}(d - a, d - b, d - c). \]

Proposition

If \( T = T_d(x^{a+\alpha}, y^b, z^c, x^a y^\beta, x^a z^\gamma) \) is balanced, then

\[ |\det Z(T)| = \text{perm } Z(T) \]
\[ = \text{Mac}(d - a, d - b, d - c) \text{Mac}(d - a - \alpha, d - a - \beta, d - a - \gamma). \]
Proposition

If $T = T_d(x^a, y^b, z^c, x^\alpha y^\beta)$ is balanced (as below), then

$$|\det Z(T)| = \text{perm} Z(T)$$

is

$$\text{Mac}(a+\beta-d, d-a, d-(\alpha+\beta)) \text{ Mac}(\alpha+b-d, d-b, d-(\alpha+\beta))$$

$$\times \frac{\mathcal{H}(d-a+d-(\alpha+\beta))\mathcal{H}(d-b+d-(\alpha+\beta))\mathcal{H}(d-c+d-(\alpha+\beta))\mathcal{H}(d)}{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(d-(\alpha+\beta))}.$$
$I \subset R = K[x, y, z]$ artinian monomial ideal.

If $K$ is infinite, then $R/I$ has the WLP iff multiplications by
$
\ell = x + y + z
$
have maximal rank.

Theorem
For each $d \geq 1$, the coordinate matrix of
$R/I^d - 2x + y + z \to R/I^{d-1}$
with respect to monomial bases in
$\text{revlex order}$ is $Z(T_d(I))$.

$\dim K[R/(I, x+y+z)]_{d-1} = \dim K(\ker N(T_d(I)))$.

Corollary
Assume $T = T_d(I)$ is balanced and the socle elements of
$R/I$ have degrees $\geq d-1$. TFAE:
$R/I$ has the WLP.

$\det Z(T_d(I)) \mod (\text{char } K) \neq 0$.

$\det N(T_d(I)) \mod (\text{char } K) \neq 0$. 
Relation to WLP

$I \subset R = K[x, y, z]$ artinian monomial ideal.

If $K$ is infinite, then $R/I$ has the WLP iff multiplications by $\ell = x + y + z$ have maximal rank.

**Theorem**

For each $d \geq 1$, the coordinate matrix of $[R/I]_{d-2}^{x+y+z} \rightarrow [R/I]_{d-1}$ with respect to monomial bases in revlex order is $Z(T_d(I))$.

$\dim_K[R/(I, x + y + z)]_{d-1} = \dim_K(\ker N(T_d(I))^T)$. 
Relation to WLP

\[ I \subset R = K[x, y, z] \] artinian monomial ideal.

If \( K \) is infinite, then \( R/I \) has the WLP iff multiplications by 
\( \ell = x + y + z \) have maximal rank.

**Theorem**

For each \( d \geq 1 \), the coordinate matrix of 
\[ [R/I]_{d-2}^{x+y+z} \rightarrow [R/I]_{d-1} \] with respect to monomial bases in revlex order is \( Z(T_d(I)) \).

\[ \dim_K[R/(I, x + y + z)]_{d-1} = \dim_K(\ker N(T_d(I))^T). \]

**Corollary**

Assume \( T = T_d(I) \) is balanced and the socle elements of \( R/I \) have degrees \( \geq d - 1 \). TFAE:

- \( R/I \) has the WLP.
- \( \det Z(T_d(I)) \mod (\text{char } K) \neq 0. \)
- \( \det N(T_d(I)) \mod (\text{char } K) \neq 0. \)
Proposition

If $R/I$ has type two, then $I$ has one of the following two forms:

(i) $I = (x^a, y^b, z^c, x^\alpha y^\beta)$,

(ii) $I = (x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$,

where $0 < \alpha < a$, $0 < \beta < b$, and $0 < \gamma < c$. 

$$T_d(x^a, y^b, z^c, x^\alpha y^\beta)$$

$$T_d(x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$$
Type two algebras

Assume char $K = 0$.

**Theorem**

If $R/I$ has type two, then $R/I$ fails to have the WLP if and only if $I = (x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$ and there exists an integer $d$ such that

$$\max \left\{a, \alpha + \beta, \alpha + \gamma, \frac{a + \alpha + \beta + \gamma}{2} \right\} < d$$

$$< \min \left\{a + \beta + \gamma, \frac{\alpha + b + c}{2}, b + c, \alpha + c, \alpha + b \right\}.$$

**Corollary (BMMNZ, 2012)**

If $R/I$ has type two and is level, then $R/I$ has the WLP.
Type two algebras

Assume \( \text{char } K = 0 \).

Theorem

If \( R/I \) has type two, then \( R/I \) fails to have the WLP if and only if \( I = (x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma) \) and there exists an integer \( d \) such that

\[
\max \left\{ a, \alpha + \beta, \alpha + \gamma, \frac{a + \alpha + \beta + \gamma}{2} \right\} < d
\]

\[
< \min \left\{ a + \beta + \gamma, \frac{\alpha + b + c}{2}, b + c, \alpha + c, \alpha + b \right\}.
\]

Corollary (BMMNZ, 2012)

If \( R/I \) has type two and is level, then \( R/I \) has the WLP.

New proof: \( R/I \) is level if and only if \( a - \alpha = b - \beta + c - \gamma \). Then

\[
\frac{a + \alpha + \beta + \gamma}{2} = \frac{2\alpha + b + c}{2} \geq \alpha + \min\{b, c\}.
\]
Proof of the Theorem (sketch)

Decompose $T_d(I)$:

$T_d(x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$

$T^u = T_{d-\alpha}(x^{a-\alpha}, y^\beta, z^\gamma)$

$T^l = T_d(x^\alpha, y^b, z^c)$
Proof (sketch)

\[
\begin{array}{ccc}
\bigtriangleup & \text{Balanced} & \bigtriangledown \\
\hline \\
\bigtriangleup & 1 & 2 & 9 \\
\hline \\
\text{Balanced} & 3 & 4 & 5 \\
\hline \\
\bigtriangledown & 8 & 6 & 7 \\
\end{array}
\]
Cases 1 – 7:

$T^u = (x^4, y^4, z^5)$

$\# \triangle = 14$

$\# \square = 13$

$T^l = (x^3, y^8, z^8)$

$\# \triangle = 22$

$\# \square = 21$

$T_{10}(x^7, y^8, z^8, x^3 y^4, x^3 z^5)$
Cases 1 – 7:

\[ T^u = (x^4, y^4, z^5) \]
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Case 8:

$T_u = (x^5, y^5, z^6)$

$\# \nabla = 19$

$\# \triangle = 21$

$T_l = (x^3, y^8, z^8)$

$\# \nabla = 22$

$\# \triangle = 21$

$T_{10}(x^8, y^8, z^8, x^3 y^5, x^3 z^6)$
Case 8:

$T^u = (x^5, y^5, z^6)$

$\# \updownarrow = 19$

$\# \triangle = 20$

$T^l = (x^3, y^8, z^8)$

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$\# \triangle = 21$

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Case 8:

\[ T^u = (x^5, y^5, z^6) \]
\[ \#\triangle = 19 \]

\[ T^l = (x^3, y^8, z^8) \]
\[ \#\triangle = 21 \]

\[ T_{10}(x^8, y^8, z^8, x^3y^5, x^3z^6) \]
Proof (sketch)

Case 9:

$T^u = (x^4, y^4, z^5)$

$\#\nabla = 14$

$\#\triangle = 13$

$T^l = (x^3, y^9, z^9)$

$\#\nabla = 24$

$\#\triangle = 25$

$T_{10}(x^7, y^9, z^9, x^3 y^4, x^3 z^5)$
$X \subset \mathbb{P}^N = \mathbb{P}^N_K \; n$-dim proj. variety, $K = \overline{K}$, char $K = 0$

$P \in X$ a smooth point, $\varphi$ a local parametrization around $P$

$T^{(s)}_P X = \mathbb{P}(\text{span of partial derivatives of } \varphi \text{ of order at most } s)$

$s$-th osculating space to $X$ at $P$

Expected dimension is $\left( \binom{n+s}{s} \right) - 1$. 

Togliatti, 1929, 1946
Perkinson, 2000
Mezzetti, Miró-Roig, Ottaviani, 2012
Di Genaro, Ilardi, Vallès, 2012
Laplace equations

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$P \in X$ a smooth point, $\varphi$ a local parametrization around $P$

$T_P^{(s)} X = \mathbb{P}$(span of partial derivatives of $\varphi$ of order at most $s$)

$s$-th osculating space to $X$ at $P$

Expected dimension is $\binom{n+s}{s} - 1$.

Definition

$X$ is said to satisfy $\delta$ Laplace equations of order $s$ if, for a general point $P$ of $X$,

$$\dim T_P^{(s)} X = \left( \binom{n+s}{s} \right) - 1 - \delta.$$

Interesting only if $N \geq \left( \binom{n+s}{s} \right) - 1$.

Togliatti, 1929, 1946
Perkinson, 2000
Mezzetti, Miró-Roig, Ottaviani, 2012
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Laplace equations

\( I = (f_1, \ldots, f_r) \subset S = K[x_0, \ldots, x_n], \text{ where } \deg f_i = d \)

\( \varphi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{r-1} \) induced rational map with image \( X_{n,[I]_d} \)
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**Example (Togliatti)**

Let \( n = 2, \quad J = (x^2y, x^2z, xy^2, xz^2, y^2z, yz^2) \). Then \( X_{2,[J]_3} \subset \mathbb{P}^5 \) is a toric surface satisfying one Laplace equation of order 2.
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\( I^{-1} \) inverse system of \( I \)

\[ \varphi_{I^{-1}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{n}-r-1} \text{ induced rational map with image } X_{n,[I^{-1}]_d} \]
Laplace equations

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\( I^{-1} \) inverse system of \( I \)

\( \varphi_{I^{-1}} : \mathbb{P}^n \to \mathbb{P}^{(n+d)/n-1} \) induced rational map with image \( X_{n,[I^{-1}]_d} \)

Remark

(i) If \( I \subset S \) is an artinian monomial ideal, then \( I^{-1} \) is generated by
monomials in \( S \setminus I \).
(ii) \( \dim_K [I^{-1}]_d = \dim_K [S/I]_d \).
(iii) If \( I = (x^3, y^3, z^3, xyz) \), then \( I^{-1} = J \).
Laplace equations

Mezzetti, Miró-Roig, Ottaviani, 2012: connection to WLP

Theorem

$I \subset S$ artinian ideal with $r \leq \binom{n+d}{n}$ minimal generators of degree $d$, $\ell \in [S]_1$ general. TFAE:

(a) Multiplication map $[S/I]_{d-1} \xrightarrow{\ell} [S/I]_d$ has a $\delta$-dim kernel.

(b) $X_{n,[I^{-1}]_d} = \varphi_{I^{-1}}(\mathbb{P}^n)$ satisfies $\delta$ Laplace equations of order $d - 1$.

If $\delta > 0$, then $I$ is said to define a Togliatti system.
Laplace equations

Mezzetti, Miró-Roig, Ottaviani, 2012: connection to WLP

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If $\delta > 0$, then $I$ is said to define a **Togliatti system**.

Assume $n = 2$, $I \subset R = K[x, y, z]$ monomial.

**Example**

Togliatti systems with few generators:

(i) (Franco, Ilardi, 2002; Vallès, 2006) 4 generators:

$$I = (x^3, y^3, z^3, xyz).$$

(ii) 5 generators: $I = (x^4, y^4, z^4, x^2yz, y^2z^2)$ or

$$I = (x^d, y^d, z^d, x^{d-1}y, x^{d-1}z).$$
Proposition

Let $U \subset T_{d+1}(I)$ be a tileable monomial subregion such that $\det Z(U) \neq 0$. Let $J$ be a monomial ideal such that $T \setminus U = T_{d+1}(J)$.

Then $[R/I]_{d-1} \xrightarrow{x+y+z} [R/I]_d$ and $[R/J]_{d-1} \xrightarrow{x+y+z} [R/J]_d$ fail to have maximal rank by the same margin.
Proposition

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Example

Togliatti systems obtained from $T_6(x^5, y^5, z^5, xyz)$.

(x^5, y^5, z^5, xyz^3, xy^3z, x^3yz) (x^5, y^5, z^5, xy^2z^2, x^2yz^2, x^2y^2z)
Proposition

Assume, \([R/I]_{d-1} \xrightarrow{x+y+z} [R/I]_d\) is not injective although it is expected
\((\dim_K [R/I]_{d-1} \leq \dim_K [R/I]_d)\), \(T = T_{d+1}(I)\) has no overlapping
punctures, and \(x^d, y^d, z^d \in I\).

For each puncture, in each row fill in all triangles, but one \(\triangle\)-triangle. Call the result \(T'\), and let \(J\) be the smallest ideal such that
\(T' = T_{d+1}(J)\). Then \(J\) defines a Togliatti system.
**Theorem**

Let $j$ be an integer such that $1 \leq j \leq \frac{d-1}{4}$ and

$$I_j = (y^d) + z^{4j+1}(y, z)^{d-1-4j} + (x^3, y^3, z^3, xyz) \cdot x^{d+1-4j} \cdot (x^4, y^4)^{j-1}.$$

Then:

(a) $[R/I_j]_{k-1} \xrightarrow{x+y+z} [R/I]_k$ has maximal rank for all $k \neq d$.

(b) $Z_{d+1}(I_j)$ is balanced.

(c) $X_{n,[(I_j)^{-1}]}_d$ satisfies exactly $j$ Laplace equations of order $d - 1$. 
Laplace equations

Example

$T_{14}(I_2)$