NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross   Email/Phone: egross7@uic.edu

Speaker's Name: Steven Sam

Talk Title: Homology of Littlewood complexes

Date: 12/7/12   Time: 9:00 am/ pm (circle one)

List 6-12 key words for the talk: Littlewood complexes, minimal free resolutions, determinant variety, Schur functions, symplectic group, invariant theory, koszul homology

Please summarize the lecture in 5 or fewer sentences:
Introduces Littlewood complexes & gives an algorithm for computing their homology groups. Explains how the homology groups of Littlewood complexes are connected to the minimal free resolutions of determinantal ideals and Koszul homology.

CHECK LIST

(This is NOT optional, we will not pay for incomplete forms)

☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  • Computer Presentations: Obtain a copy of their presentation
  • Overhead: Obtain a copy or use the originals and scan them
  • Blackboard: Take blackboard notes in black or blue PEN. We will NOT accept notes in pencil or in colored ink other than black or blue.
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  (YYYY.MM.DD.TIME.SpeakerLastName)

☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.
Homology of Littlewood Complexes

joint w/ A. Snowden, J. Weyman

0. Motivation: Boij - Söderberg theory for asymptotic quadrics

1. Symmetric functions

2. Classical invariant theory

3. Resolutions of modules supported in determinental varieties.

Eisenbud - Fløystad - Weyman:

Construction of "pure free resolutions" over polynomial rings using rep theory of GL(n).

Long-term goal. Extend construction for homogeneous coordinate ring of a smooth quadric using representation theory of orthogonal group $O(n)$

1. Symmetric Functions

Representation Theory of $GL(n)$

$\leftrightarrow$ symmetric polynomials in $n$ variables

$s_n \rightarrow \infty$

ring of symmetric functions
(for talk: focus on rep theory of $\text{Sp}(2k)$)

Symplectic group character ring

\[ \kappa \rightarrow \infty \quad (\text{Koike-Terada}) \]

ring of symmetric functions $\Lambda$

Important structure: specialization maps

$\Lambda \rightarrow \begin{cases} 
\text{char ring of } \text{GL}(n) \\
\text{char ring of } \text{Sp}(2k)
\end{cases}$

- Distinguished basis for $\Lambda$: for $\text{GL}(n)$
  $\nu \mapsto$ Schur functions $s_{\nu}$, $\lambda$ partition
  for $\text{Sp}(2k) \mapsto s_{\nu_{\lambda}}$, $\lambda$ partition

Idea: branching from $\text{GL}(2k)$ to $\text{Sp}(2k)$

For a given irreducible, indexed by $\lambda$

is independent of if $k \gg 0$ w.r.t $\lambda$

The change of basis for this branching rule is upper triangular

One way to encode this:

Littlewood complexes

$\begin{cases} 
\text{$\nu_{\lambda}$ resolution} \rightarrow s_{\lambda} \rightarrow s_{\nu_{\lambda}} \rightarrow 0 
\end{cases}$

by $s_{\mu}, \mu_{1} < \lambda$
These $L^\lambda$ were constructed for $k \gg 0$ but they make sense for any $k$.

Problem: What is homology of $L^\lambda$. For any $k$?

Punchline: $L^\lambda$ has homology is at most one degree & homology is irreducible representation if it exists.

Rule for calculating $H_0(L^\lambda)$

1. If $\lambda$ has at most $k$ rows, then $H_0(L^\lambda) = S[\lambda]$.

2. Set $r = 2(l(\lambda) - k - 1)$

Try to remove border strip of length $r$ starting from the first box in the last row. Go back to step 1.

Conclusion: if step 2 fails, then $L^\lambda$ is exact, otherwise $H_{\ast}(L^\lambda) = S[\mu]$ where $\mu$ satisfies condition 1.

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| $\nu = 2$, $Sp(4)$ | $l(\lambda = 7)$

![Diagram](image-url)
2. Classical invariant theory

\[ V = \mathbb{C}^{2^k} \text{ vector rep of } \text{Sp}(2k) \]
\[ E = \mathbb{C}^n \text{ auxiliary vector space} \]

Let \( X = \text{Hom}(E, V) \cong V \otimes E \)

Weyl : ring \( \mathbb{R} \) of invariants of \( X \)

is generated

given any two vectors \( e, e' \in E \) \& \( \phi \in \mathfrak{X}_E \)

function \( (\phi(e), \phi(e')) \)

\( \in \text{symplectic pairing on } V \)

These functions generate ring of invariants

\( (\mathfrak{X}_E) \) functions

\( \text{ideal } \mathfrak{I} \)

Variety cut out by invariants \( \phi \) of symplectic

form vanishes on image \( (\phi) \)

Fact : If \( k > n \), then this ideal defines a

complete intersection:

Resolution of \( \mathbb{C}[x] / \mathfrak{I} \)

\[ 0 \leftarrow A / \mathfrak{I} \leftarrow A \leftarrow A \otimes \mathfrak{X}_E \leftarrow A \otimes \mathfrak{X}_E \leftarrow A \otimes \mathfrak{X}_E \]

\( \leftarrow A \otimes \mathfrak{X}_E \leftarrow \cdots \leftarrow \)
Note: \( GL(E) \) also acts on \( X \), Koszul complex gives an isotypic decomposition for Koszul complex.

Fix representation \( S^\lambda(E) \) of \( GL(E) \).

\[ \text{Hom}_{GL(E)}(S^\lambda E, \text{Koszul complex}) \leftrightarrow \lambda \text{ isotypic component} \]

Important point

\[ A \overset{\bigoplus \phi}{\longrightarrow} GL(E)^{\times \text{Sp}(V)} \]

Punchline: \( L^\lambda \) gives resolution of \( S^\lambda \psi V \) in terms of thing of the form \( S^\mu V \).

Problem of calculating \( H_0(L^\lambda) \) translates to "what is Koszul homology of \( I \) w/ no assumption on \( n \)?" in general, complicated

One more rephrasing:

- subring of \( \text{Sp}(2k) \)-invariants has a nice interpretation - it cuts out a determinantal variety

\[ X = \text{Hom}(E, V) \]

\[ E \twoheadrightarrow V \cong V^* \xrightarrow{\phi} E^* \]

\( \phi \xrightarrow{\phi^*} \phi^* \) realsizes quotient by \( \text{Sp}(2k) \) skew-symmetric of rank \( k \).
image is defined by Pfaffians of size \(2k+2\)

\(C^J\) is a module over space of skew-symmetric matrices.

Reformulation of problem calculate minimal free resolution of this module

\(C^J\) is not f.g. module

To fix this: decompose \(C^J = \bigoplus_\lambda M_\lambda \otimes S_{C^J}(\nu)\)

then \(M_\nu = \text{dtl. ring, each } M_\lambda \text{ is f.g. over it}

**Thm**

\[
\text{Tor}_i^{\text{Sym}(A^\ast E)}(M_\lambda, C) = \bigoplus_\mu S_{\lambda \times E}^\mu,
\]

\(\lambda(\mu) \leq n = \dim E\)

\(M \rightarrow \lambda \text{ w/ i total columns}

\(\uparrow \text{ rule explained earlier}

\[\phi \leftarrow \begin{array}{c}
\end{array}\]

\[\begin{array}{c}
\end{array}\]

Buchsbaum-Eisenbud resolution

\[\begin{array}{c}
\end{array}\]