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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross **Email/Phone:** egross7@uic.edu

Speaker's Name: Steven Sam

Talk Title: Homology of Littlewood complexes

Date: 12 / 7 / 12 **Time:** 9 : 00 am / pm (circle one)

List 6-12 key words for the talk: Littlewood complexes, minimal free resolutions, determinantal variety, Schur functions, symplectic group, invariant theory, Koszul homology

Please summarize the lecture in 5 or fewer sentances: Introduces Littlewood complexes & gives an algorithm for computing their homology groups. Explains how the homology groups of Littlewood complexes are connected to the minimal free resolutions of determinantal ideals and Koszul homology.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Homology of Littlewood Complexes

joint w/ A. Snowden, J. Weyman

0. Motivation: Boij - Söderberg theory for ~~quadrics~~ quadrics

1. Symmetric functions

2. Classical invariant theory

3. Resolutions of modules supported in determinental varieties.

Eisenbud - Fløystad - Weyman:

Construction of "pure free resolutions" over polynomial rings using rep theory of $GL(n)$.

Long-term goal. Extend construction for homogeneous coordinate ring of a smooth quadric using representation theory of orthogonal group gp $O(n)$

1. Symmetric Functions

Representation Theory of $GL(n)$

$\leftrightarrow$ symmetric polynomials in n variables

$\downarrow$ $n \to \infty$

ring of symmetric functions

(for talk: focus on rep theory of $Sp(2k)$)

symplectic group | character ring
$\downarrow$ $k \to \infty$ (Koike-Terada)
ring of symmetric functions Λ

<u>important structure</u>: specialization maps

$$\Lambda \to \begin{cases} \text{char ring of } GL(n) \\ \text{char ring of } Sp(2k) \end{cases}$$

- distinguished basis for Λ: for $GL(n)$
  $\rightsquigarrow$ Schur functions s_λ, λ partition

for $Sp(2k)$ $\rightsquigarrow$ $s_{[\lambda]}$, λ partition

<u>Idea</u> branching from $GL(2k)$ to $Sp(2k)$ for a given irreducible, indexed by λ is independent of if $k \gg 0$ wrt λ & the change of basis for this branching rule is upper triangular

One way to encode this:

Littlewood complexes $L_\bullet^\lambda$ $\begin{cases} \exists \text{ resolution} \\ \text{by } S_\mu, |\mu| < |\lambda| \end{cases} \to S_\lambda \twoheadrightarrow S_{[\lambda]} \to 0$

③

These $L^\lambda_\bullet$ were constructed for $k >> 0$ but they make sense for any k.

Problem What is homology of L^λ. For any k?

Punchline $L^\lambda_\bullet$ has homology is at most one degree & homology is irreducible representation if it exists.

Rule for calculating $H_\bullet(L^\lambda_\bullet)$

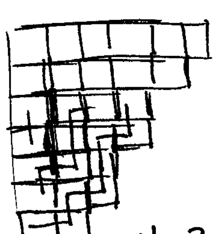
1. If λ has at most k rows $\xleftarrow{} \ell(\lambda)$, then $H_0(L^\lambda_\bullet) = S_{[\lambda]}$

2. Set $r = 2(\ell(\lambda) - k - 1)$

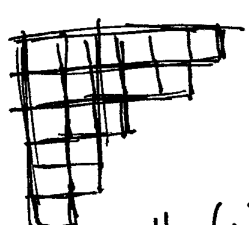
   Try to remove border strip of length r starting from the first box in the last row. Go back to step 1

Conclusion: if step 2 fails, then $L^\lambda_\bullet$ is exact otherwise $H_{i(\lambda)}(L^\lambda_\bullet) = S_{[\mu]}$ where μ satisfies condition 1.

6 5 4 4 3 3 2 , $k = 2$, $Sp(4)$ $\ell(\lambda = 7$



after $\rightsquigarrow$ step 2



$H_8(L^\lambda_\bullet) = S_{[6,5]}$

2. Classical invariant theory

$V = \mathbb{C}^{2k}$ vector rep of $Sp(2k)$

$E = \mathbb{C}^n$ auxiliary vector space

Let $X = \mathrm{Hom}(E, V) \cong V^{\oplus n}$

Weyl: ring ~~in~~ of invariants of X is generated

given any two vectors $e, e' \in E$ & $\phi \in X \rightsquigarrow$ function $(\phi(e), \phi(e'))$

symplectic pairing on V

These functions generate ring of invariants

$\binom{n}{2}$ functions

ideal I

Variety cut out by invariants $\{\phi \mid$ symplectic form vanishes on image $(\phi)\}$

Fact If $k \geq n$, then this ideal defines a complete intersection:

Resolution of $\underbrace{\mathbb{C}[X]}_{A}/I$

$$0 \leftarrow A/I \leftarrow A \leftarrow A \otimes \Lambda^2 E \leftarrow A \otimes \Lambda^2(\Lambda^2 E)$$
$$\leftarrow A \otimes \Lambda^3(\Lambda^2 E) \leftarrow \cdots \leftarrow$$

Note: $GL(E)$ also acts on X, Koszul complex gives an isotypic decomposition for Koszul complex

Fix representation $S_\lambda(E)$ of $GL(E)$.

$\mathrm{Hom}_{GL(E)}(S_\lambda E, \text{Koszul complex}) \leftarrow$ λ isotypic component $L_\bullet^\lambda$

Important point

$$A \backslash I \overset{GL(E)\times Sp(V)}{=\!=} \bigoplus_{\ell(\lambda)\le n} S_\lambda(E) \otimes S_{[\lambda]}(V)$$

Punchline: $L_\bullet^\lambda$ gives resolution of $S_{[\lambda]} V$ in terms of thing of the form $S_\mu V$

Problem of calculating $H_\bullet(L_\bullet^\lambda)$ translates to "what is $\underbrace{\text{Koszul homology}}$ of I w/ no assumption on n?" ← in general, complicated

One more rephrasing:

- subring of $Sp(2k)$-invariants has a nice interpretation - it cuts out a determinantal variety

$X = \mathrm{Hom}(E, V)$ $\quad E \to V \overset{\text{symplectic form}}{\cong} V^* \xrightarrow{\phi^*} E^*$

$\phi \in X$ $\quad \phi \longmapsto \phi^*\phi$ realizes quotient by $Sp(2k)$ ← skew-symmetric of rank $\le 2k$

image is defined by Pfaffians of size $2k+2$

$\mathbb{C}[x]$ is a module over space of skew-symmetric matrices.

<u>Reformulation of problem</u> calculate ~~min~~ min free resolution of this module

$\mathbb{C}[x]$ is not f.g. module

<u>To fix this</u>: decompose $\mathbb{C}[x] = \bigoplus_\lambda M_\lambda \otimes S_{[\lambda]}(V)$

then $M_\emptyset$ = dut'l ring, each M_λ^λ is f.g. over it

<u>Thm</u>

$$\mathrm{Tor}_i^{\mathrm{Sym}(\Lambda^2 E)}(M_\lambda, \mathbb{C})$$

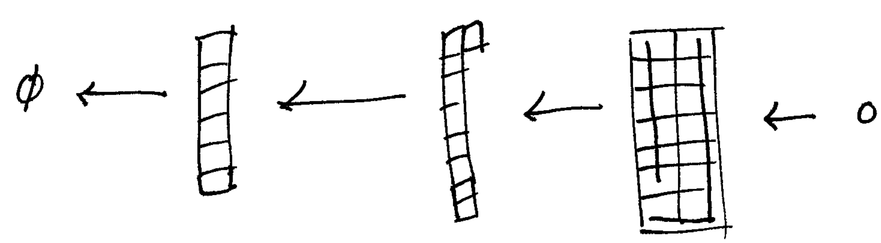
$$= \bigoplus_\mu S_\lambda E$$

$\ell(\mu) \le n = \dim E$

$\mu \longrightarrow \lambda$ w/ i total columns

rule explained earlier

$M_\emptyset$



$k=2$

$n=7$

rank ≤ 4

skew 7×7 mat

Buchsbaum-Eisenbud resolution

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