What does a Point Process Outside a Domain tell us about What's Inside ?

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Key Characteristics:

- Independence between spatially disjoint domains
- In any given domain D, the number of points N follows Poisson distribution , with parameter $\propto$ area(D)
- The N points are distributed uniformly in D and independently of each other

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- Arise in physics as mathematical models

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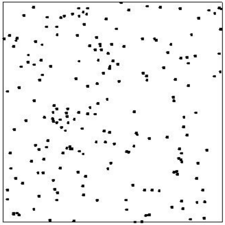

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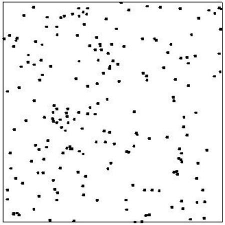

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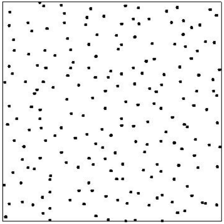

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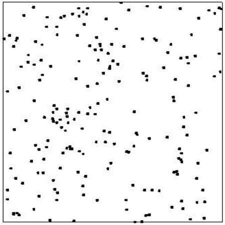

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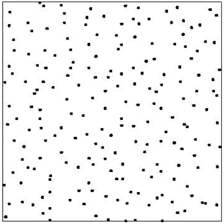

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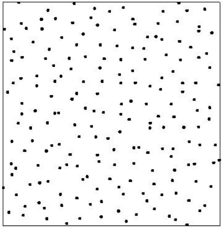

Gaussian Zeroes

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Convergence of Point Processes:

- $\mu_n \xrightarrow{d} \mu$ iff $\int \varphi d\mu_n \xrightarrow{d} \int \varphi d\mu \quad \forall \varphi \in C_c(\mathbb{C})$

Ginibre Ensemble

- Finite n: $\mu_n =$ Eigenvalues of $G_n = ((\xi_{ij}))_{1\leq i,j\leq n}$, ξ_{ij} i.i.d $N_{\mathbb{C}}(0,1)$ (NO normalization by $\sqrt{n}$)

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Theorem (Sodin rigidity)

$f(z)$ is the unique (up to a deterministic multiplier) Gaussian entire function with a translation invariant zero process of intensity 1.

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  - Holroyd and Soo (2010)

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- In Poisson point process, the points inside and outside $\mathbb{D}$ are independent of each other

Our Results: Ginibre Ensemble

Theorem (Ghosh,Nazarov,P., Sodin,'12)

In the Ginibre ensemble,
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$M(\omega)$ and $m(\omega)$ positive constants
$\Delta(\zeta) = \prod_{i<j}(\zeta_i - \zeta_j)$ (Vandermonde)

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Limit as $n \to \infty$ is Poisson : Not Rigid !

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$\Sigma_{S(\omega)}$: constant sum hypersurface $\sum_{i=1}^{N(\omega)} \zeta_i = S(\omega)$ inside $\mathbb{D}^{N(\omega)}$

$M(\omega)$, $m(\omega)$ and $\Delta(\zeta)$ are as before.

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- Application to continuum percolation

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  Want: number of inside zeroes
- Linear Statistic: $\int \varphi d\nu, \varphi \in C_c^2(\mathbb{C})$
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- Know outside zeroes $\Rightarrow$ Know $\int_{\mathbb{D}_L \setminus \mathbb{D}} \varphi_L d\nu \Rightarrow$ Compute $n(\mathbb{D})$ approximately, now let $L \to \infty$

An Intriguing Consequence of our Estimates

Proposition (Reconstruction of Gaussian Analytic Function)

*The zeroes of the GAF determine the function a.s. (up to a multiplicative factor of modulus 1). In other words, if ν denotes the zeroes of the GAF f, then $\exists$ an analytic function $g(z) = \sum_{k=0}^{\infty} a_k(\nu) z^k$ such that $f(z) = \gamma . g(z)$
Here γ follows $Unif(S^1)$ and is independent of ν.*