- Speaker:Marek Biskup (joint work with Salvi and Wolff)
- Title:A central limit theorem for the effective conductance and resistance
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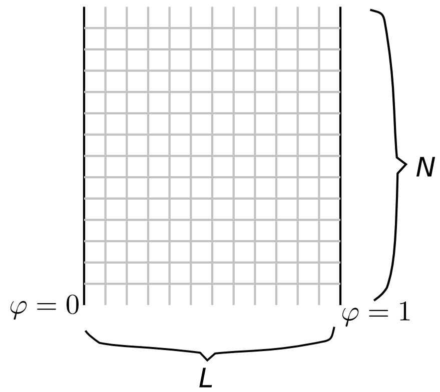
1. Model. Place the lattice inside two metal plates (see graph). Each edge has an conductance C_{xy}. Then the effective conductance on the



lattice $\Lambda_{N,L}$ is

$$R_{N,L}^{-1} = \inf_{f|\partial\Lambda_{N,L}=\varphi} Q_{\Lambda_{N,L}}(f) = O(N^{d-1}/L) \overset{\text{If } N=L}{=} O(N^{d-2}).$$

Here for a finite set $\Lambda \subset \mathbb{Z}^d, d \geq 2$,

$$Q_\Lambda(f) = \sum_{(x,y)\in B(\Lambda)} C_{xy}(f(x) - f(y))^2.$$

$B(\Lambda)$: edges with one endpoint in Λ.
$C_{xy} = C_{yx} \in (0, \infty)$: conductance.
$r_{xy} = 1/C_{xy}$.

2. Homogenization: whenever (C_{xy}) is stationary ergodic and $EC_{xy} < \infty$,

$$\lim_{N\to\infty} \inf_{f|\partial\Lambda_N=\varphi} \inf Q_N(f)/N^{d-2} = \inf_{f|\partial\Lambda_1=\varphi} \int_{\Lambda\subset\mathbb{R}^d} \sum_{i,j=1}^{d} \hat{C_{ij}} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \,\mathrm{d}x$$

almost surely. (Ref: Jikov-Kozlov-Oleinik)

3. Simplest nontrivial problem:
   - linear boundary condition: $\varphi(x) = t \cdot x$.

- cubic box

Subadditive ergodic theorem$\Longrightarrow \lim_{N\to\infty} \inf_{f|_{\partial\Lambda}=t\cdot x} \frac{Q_{\Lambda_N}(f)}{N^d}$ exists.(Kunnemann)

Theorem(B.-Salvi-Wolff)

Suppose $d \geq 2$, $(C_{xy})_{x\sim y}$ is iid and elliptic with small ellipticity contrast. Then

$$\left(\inf_{f|_{\partial\Lambda_N}=t\cdot x} Q_{\Lambda_N}(f) - E \inf_{f|_{\partial\Lambda_N}=t\cdot x} Q_{\Lambda_N}(f) \right) / |\Lambda_N|^{1/2}$$
$$\Longrightarrow \mathcal{N}(0, \sigma_t^2) \qquad \text{as } N \to \infty.$$

$\sigma_t^2 \in (0, \infty)$ when $t \neq 0$.

(Related works: Benjamini-Rossignol (Wehr lower bound), Gloria-Otto)

4. Homogenization again.
   Let the operator ("random Laplacian")
   $$Lf(x) := \sum_{y:y\sim x} C_{xy}[f(y) - f(x)].$$
   Fact: $\inf_{f|_{\partial\Lambda}=t\cdot x} Q_\Lambda(f)$ is achieved at $f(x) = t \cdot \Psi_\Lambda(x)$, where $\Psi_\Lambda(x)$ satisfies
   $$\begin{cases} L\Psi_\Lambda(x) = 0, & \forall x \in \Lambda \\ \Psi_\Lambda(x) = x, & x \in \partial\Lambda. \end{cases}$$
   We need to find $\psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ such that:
   (1) $L\psi(\omega, \cdot) = 0$;
   (2) $\{\psi(\omega, x+z) - \psi(\omega, x)\}_{z\in\mathbb{Z}^d}$ is stationary.
   (3) $\psi(\omega, x+z) - \psi(\omega, x) \in L^2(\mathbb{R})$ for all z.
   (4) $\psi(\omega, x) - x = o(|x|)$ as $|x| \to \infty$.
   Look for functions of type $\psi(\omega, x) = x + \nabla_x\phi(\omega)$, where $\nabla_x\phi(\omega) := \phi(\tau_x\omega) - \phi(\omega)$.

5. Limiting effective resistance
   Idea: replace Ψ_Λ by ψ inside the Dirichlet energy.
   Lemma 1 $|Q_\Lambda(f+h) - Q_\Lambda(f)| \leq Q_\Lambda(h) + 2Q_\Lambda(f)^{\frac{1}{2}} Q_\Lambda(h)^{\frac{1}{2}}$.
   Lemma 2. Suppose $Lh = 0$ in Λ. Then
   $$Q_\Lambda(h) = \frac{1}{2} \sum_{x,y\in\partial\Lambda} K_\Lambda(x,y)(h(x) - h(y))^2,$$
   where $K_\Lambda(\cdot, \cdot) > 0$.

   Applying the Lemmas to $f(x) = t \cdot \Psi_\Lambda(x)$ and $h(x) = t \cdot (\psi(\omega, x) - \Psi(x))$, we get (by the ergodic theorem)
   $$\lim_{N\to\infty} \frac{Q_\Lambda(t \cdot \psi(\omega, \cdot))}{|\Lambda_N|} = E[\sum_{i=1}^d \omega_{o,e_i}(t \cdot \psi(\omega, e_i) - t \cdot \psi(\omega, o))^2].$$

6. Proof of the Gaussian fluctuation

Assume that (C_{xy}) is iid and elliptic.

Suppose we order the edges $B(\Lambda)$ as: $e(1), e(2), \ldots, e(n)$, where $n = |B(\Lambda)|$.

Define $\mathcal{F}_k := \sigma(C_{e(j)}(\omega) : j = 1, \ldots, k)$. Then

$$\begin{aligned} Q_\Lambda(t \cdot \Psi_\Lambda) - E[Q_\Lambda(t \cdot \Psi_\Lambda)] &= \sum_{k=1}^{n} E[Q_\Lambda(t \cdot \psi_\Lambda)|\mathcal{F}_k] - E[Q_\Lambda(t \cdot \psi_\Lambda)|\mathcal{F}_{k-1}] \\ &:= \sum_{k=1}^{n} Z_{\Lambda,k}, \end{aligned}$$

and $\operatorname{Var} Q_\Lambda(t \cdot \Psi_\Lambda) = \sum_{k=1}^n E Z_{\Lambda,k}^2$.

For the Gaussian limit, we need to verify the two conditions of Lindeberg-Feller:

- $\frac{1}{n} \sum_{k=1}^n E[Z_{\Lambda,k}^2 | \mathcal{F}_{k-1}] \overset{\text{in prob.}}{\longrightarrow} \sigma_t^2$.
- $\frac{1}{n} \sum_{k=1}^n E[Z_{\Lambda,k}^2 1_{|Z_{\Lambda,k}| \geq \epsilon\sqrt{n}} | \mathcal{F}_{k-1}] \overset{\text{in prob.}}{\longrightarrow} 0$ for all $\epsilon > 0$.

Computation shows:

$$Z_{\Lambda,k} = E\left[\int \mathrm{d}P(C'_{e(k)}) \int \frac{\partial Q_\Lambda(t \cdot \psi_\Lambda)}{\partial C_{e(k)}} \,\mathrm{d}C | \mathcal{F}_k\right].$$

For $e(k) = (x, y)$, $\frac{\partial Q_\Lambda}{\partial C_{e(k)}} = |t \cdot [\Psi_\Lambda(y) - \Psi_\Lambda(x)]|^2$.

Now we order the edges such that:

$$(x, i) \leq (y, j) \text{ if either } x < y \text{ or } x = y \text{ and } i \leq j,$$

then

$$Z_{\Lambda,k} = E\left[\int \mathrm{d}P(C'_{x,x+e_k}) \int_{C_{x,x+e_k}}^{C'_{x,x+e_k}} |t \cdot \psi(\tau_\omega, e_k)|^2 \middle| \mathcal{F}_{x,k}\right].$$

We can then show

$$Z_{x,k} \in L^2(P)$$

for any elliptic iid conductances when $d \geq 3$ (Gloria-Otto), and $d \geq 2$ when the conductances have small contrast (Meyers).