Chemical distance on random interlacements

Jiří Černý
University of Vienna

joint work with Serguei Popov
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Definition of Random Interlacement

Random interlacement is a 'dependent percolation model' introduced by A.-S. Sznitman (2010).

- $W^\star$ - space of doubly-infinite n.n. trajectories on $\mathbb{Z}^d$, $d \geq 3$, modulo time-shift.
- ν - a σ-finite measure on $W^\star$.
- (w_i, u_i) - cloud of labelled trajectories,
  i.e. a Poisson point process on $W^\star \times [0, \infty)$ with intensity $\nu \otimes du$
- $\mathbb{P}$ law of this process
- $\mathcal{I}^u$ - the **interlacement set**,

$$\mathcal{I}^u = \bigcup_{i:u_i \leq u} \mathrm{Range}\, w_i$$

- $\mathcal{V}^u$ - the **vacant set**

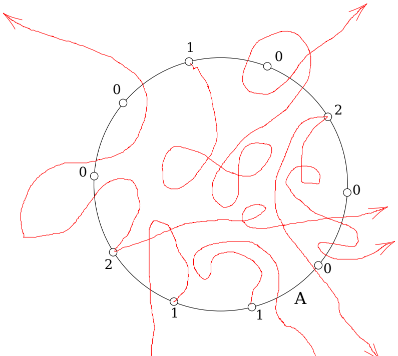
$$\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$$

Local specification for Random interlacement

Let $A \subset \mathbb{V}$ finite.

- *equilibrium measure:*
  $e_A(x) = \mathrm{Prob}[\text{RW on } \mathbb{V} \text{ started at } x \text{ never returns to } A] \cdot \mathbf{1}_A(x)$.
- for every $x \in A$, let N_x be Poisson($ue_A(x)$) random variable. N_x's independent
- at every point x start N_x independent random walks $X^{(x,i)}$, $i \le N_x$.
- Then

$$\mathcal{I}^u \cap A \overset{\text{law}}{=} A \cap \bigcup_{x \in A} \bigcup_{i \le N_x} \mathrm{Range}\, X^{(x,i)}.$$



Basic question

Understand the behaviour of the random sets $\mathcal{I}^u$ and $\mathcal{V}^u$.

Random interlacement is a correlated dependent percolation:

- density
$$\mathbb{P}[x \in \mathcal{I}^u] = 1 - e^{-u\,\mathrm{cap}(x)}$$
- correlation
$$\mathrm{Cor}_{\mathbb{P}}(x \in \mathcal{I}^u, y \in \mathcal{I}^u) \sim c(u)|x-y|^{2-d}.$$
- no duality between $\mathcal{V}^u$ and $\mathcal{I}^u$.

Phase transition for $\mathcal{V}^u$

Theorem (Sznitman '10; Sznitman, Sidoravicius '09)

For every $d \geq 3$ there is $u_\star = u_\star(d)$, such that

$$0 < u_\star < \infty$$

and

- *If $u < u_\star$, then $\mathcal{V}^u$ contains an infinite connected component $\mathbb{P}$-a.s.*
- *If $u > u_\star$, then there are $\mathbb{P}$-a.s. only finite components of $\mathcal{V}^u$.*

Absence of phase transition for $\mathcal{I}^u$

Trivially: For every $u > 0$, the interlacement set contain an infinite connected component.

Theorem (Sznitman '10)

For every $u > 0$, $d \geq 3$,

$$\mathbb{P}[\mathcal{I}^u \textit{ is connected}] = 1.$$

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Theorem (Č, Popov '12)

For every $d \geq 3$,

$$\mathbb{P}[\mathcal{I}^u \textit{ is connected for every } u > 0] = 1.$$

How connected is $\mathcal{I}^u$?

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Theorem (Procaccia, Tykesson EJP'11; Ráth, Sapozhnikov ALEA'12)

Given that $x, y \in \mathcal{I}^u$, it is possible to find a path between x and y contained in the range of at most $\lceil d/2 \rceil$ trajectories from the underlying Poisson point process.

Is $\mathcal{I}^u$ close to $\mathbb{Z}^d$?

Theorem (Ráth, Sapozhnikov ECP'11)

For every $u > 0$, $d \geq 3$, the simple random walk on $\mathcal{I}^u$ is transient.

Theorem (Ráth, Sapozhnikov arXiv:1109.5086)

Let $\mathcal{B}_p$ be the Bernoulli site percolation on $\mathbb{Z}^d$ with parameter p
There exists $p < 1$ and $R < \infty$ such that $\mathbb{P}$-a.s

$$\mathcal{I}^u \cap \mathcal{B}_p \text{ percolates in the slab } \mathbb{Z}^2 \times [-R, R]^{d-2}.$$

Chemical/graph/internal distance

Let

$$\rho_u(x,y) = \min\{n : \exists x_0, x_1, \ldots, x_n \in \mathcal{I}^u \text{ such that } x_0 = x, x_n = y, \\ \text{and } \|x_k - x_{k-1}\|_1 = 1 \text{ for all } k = 1, \ldots, n\},$$

be the graph distance on $\mathcal{I}^u$.

Question. Is it comparable to the Euclidean distance?

Large deviations for chemical distance

Let

$$\mathbb{P}_0^u[\cdot] = \mathbb{P}[\cdot | 0 \in \mathcal{I}^u].$$

Theorem (Č-Popov'12)

For every $u > 0$ and $d \geq 3$ there exist constants $C, C' < \infty$ and $\delta \in (0, 1)$ such that

$$\mathbb{P}_0^u[\textit{there exists } x \in \mathcal{I}^u \cap [-n, n]^d \textit{ such that } \rho_u(0, x) > Cn] \leq C' e^{-n^\delta}.$$

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For the Bernoulli percolation the corresponding statement was shown by Antal and Pisztora (1996), with $\delta = 1$.

We can show that $\delta = 1$ for $d \geq 5$.

The shape theorem

Let $\Lambda^u(n) = \{y \in \mathcal{I}^u : \rho_u(0,y) \leq n\}$ be the ball around 0 of radius n in the chemical distance.

Theorem

For every $u > 0$ and $d \geq 3$ there exists a compact convex set $D_u \subset \mathbb{R}^d$ such that for any $\varepsilon > 0$, $\mathbb{P}_0^u$-a.s. for n large

$$\big((1-\varepsilon)nD_u \cap \mathcal{I}^u\big) \subset \Lambda^u(n) \subset (1+\varepsilon)nD_u.$$

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Question. How D_u behaves as $u \to 0$?

Implication for RW on torus

Let X be random walk on the torus $\mathbb{T}_N^d$
and ρ_N^u the graph distance on its range $\mathcal{I}_N^u = \{X_0, \ldots, X_{uN^d}\}$.

Theorem

For large enough C and γ, we have

$$P^N\big[\rho_N^u(x,y) \le C|x-y| \,\forall x,y \in \mathcal{I}_N^u \text{ s.t. } |x-y| \ge \ln^\gamma N\big] \xrightarrow{N\to\infty} 1$$

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Improves result of Shellef(- Procaccia) arXiv:1007.1401, who shows that the same hold for $C = \underbrace{\log \ldots \log}_{k \text{ times}} N$, $k \geq 1$.

Simple proof of the large deviation result.

Works in $d \geq 5$ only!

Based on Antal-Pisztora, Liggett-Schonmann-Stacey'97, and

Lemma (Ráth-Sapozhnikov)

$$\mathbb{P}\Big[\bigcap_{x,y \in \mathcal{I}^u \cap B(R)} x \overset{B(2R)\cap \mathcal{I}^u}{\longleftrightarrow} y \Big] \geq 1 - ce^{-cR^{1/6}}.$$

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Remark. The lemma implies that with a large probability

$$\rho_u(x,y) \leq c|x-y|^d$$

Strong supercriticality of $\mathcal{I}^u$

Consider

- a box $B(n)$, $h \in [0, 2/d)$
- $\eta_1 \leq \eta_2$ such that $\eta_1 \geq n^{d-2-h}$, $\eta_2 \leq n^M$,
- η_2 independent random walks $X_k^{(i)}$ started in $B(n)$.
- ranges $R_i(m) = \{X_0^{(i)}, \dots, X_m^{(i)}\}$.

Lemma

For every $h > 0$ there is $\beta(d, h) < \infty$ such that with probability larger than $1 - ce^{-cn^{c'}}$ the following occurs:

- *Any two points in $\cup_{i \leq \eta_1} R_i(2n^2)$ can be connected by a path included in at most $\beta(h, d)$ sets $R_i(2n^2)$, $i \leq \eta_1$.*
- *For every $j \leq \eta_2$,*

$$R_j(n^2) \cap \bigcup_{i \leq \eta_1} R_i(2n^2) \neq \emptyset.$$

- *"a technical condition on remainders of trajectories".*

Technical estimates

Let $q_x(A, n)$ be the probability that the random walk started from x hits A before n, $\ell(x, A) = \max_{y \in A} |x - y|$. Then for $n \geq \ell(x, A)^2$

$$q_x(A, n) \geq \begin{cases} c \operatorname{diam}(A) \ell(x, A)^{2-d} \\ c|A|^{1-\frac{2}{d}} \ell(x, A)^{2-d} \end{cases}$$

with log corrections in $d = 3$.

The end

Thank you for your attention.