Conditional quenched CLTs for random walks among random conductances

Christophe Gallesco Nina Gantert Serguei Popov
Marina Vachkovskaia

One-dimensional random walks with unbounded jumps

Many-dimensional random walks (nearest-neighbor and uniformly elliptic)

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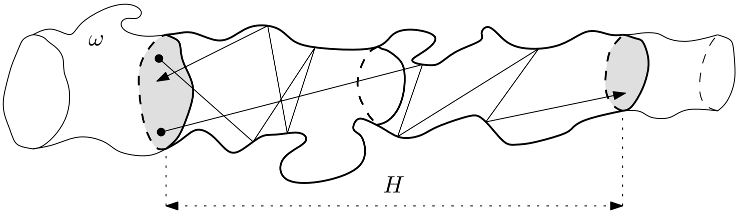

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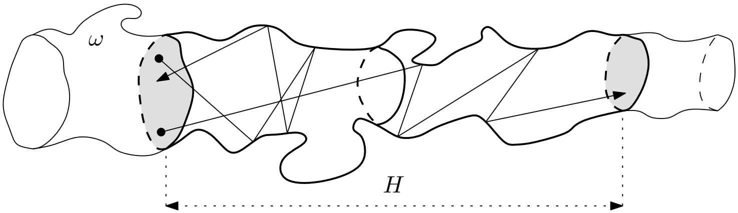

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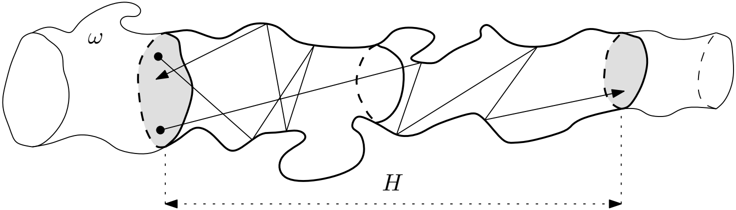

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This would be a concequence of a *conditional* CLT!

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Condition E.

(i) There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $q_\omega(0, \pm 1) \geq \kappa$.

(ii) Also, there exists $\hat{\kappa} > 0$ such that $\hat{\kappa} \leq \sum_{y \in \mathbb{Z}} \omega_{0,y} \leq \hat{\kappa}^{-1}$, $\mathbb{P}$-a.s.

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(observe that this implies that the second moment of the jump is uniformly bounded)

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Example: simple random walk S, conditioned on $\{S_1 > 0, \dots, S_n > 0\}$, after usual scaling converges to the Brownian Meander.

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For each n, the random map Z^n induces a probability measure μ^n_ω on $(C[0,1], \mathcal{B}_1)$: for any $A \in \mathcal{B}_1$,

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Main result:

Theorem

Under Conditions E and K, we have that, $\mathbb{P}$-a.s., μ_ω^n tends weakly to P_{W^+} as $n \to \infty$, where P_{W^+} is the law of the Brownian meander W^+ on $C[0,1]$.

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As a corollary of Theorem 1.1, we obtain a limit theorem for the process conditioned on crossing a large interval. Define

$$\hat{\tau}_n = \inf\{k \geq 0 : X_k \in [n, \infty)\} \quad \text{and} \quad \Lambda_n' = \{\hat{\tau}_n < \hat{\tau}\}.$$

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Corollary

Assume Conditions E and K. Then, conditioned on Λ'_n, the process converges to the "Brownian crossing".

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- the main difficulty: control the (both conditional and unconditional) exit measure from large intervals
- (observe that is ξ has only polynomial tail, then $\frac{P[x<\xi\leq x+a]}{P[\xi>x]} \to 0$ as $x \to \infty$)

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The corrector is shown to exist, but usually no explicit formula is known for it.

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Now, we formulate our main result:

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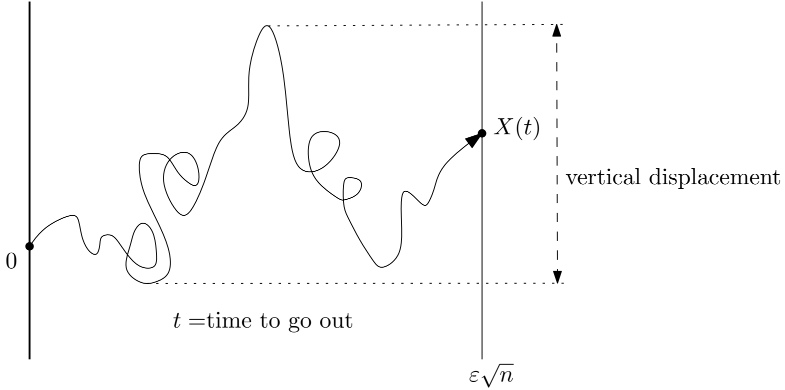

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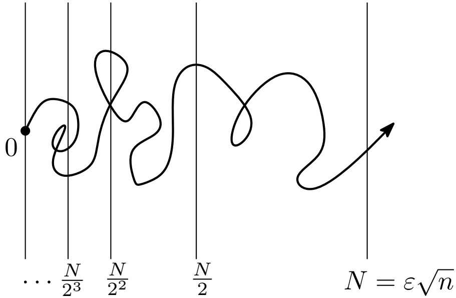

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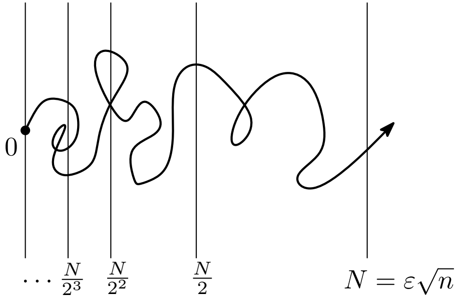

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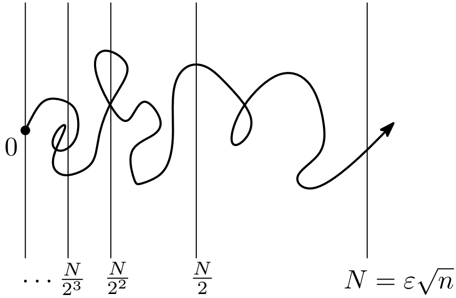

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then iterate:

$$P_\omega[\tau_{2^{-j}N} > \alpha^j n \mid \Lambda_n] \le P_\omega[\tau_{2^{-(j+1)}N} > \alpha^{j+1} n \mid \Lambda_n] + \text{smth very small}$$

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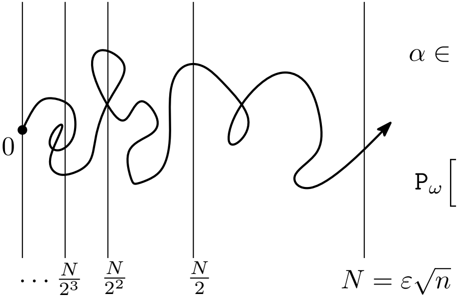

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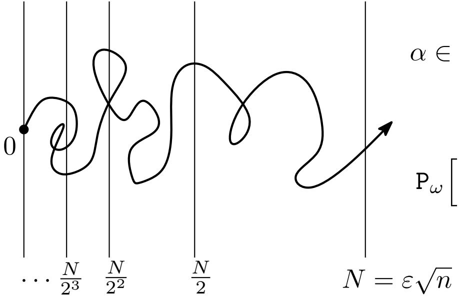

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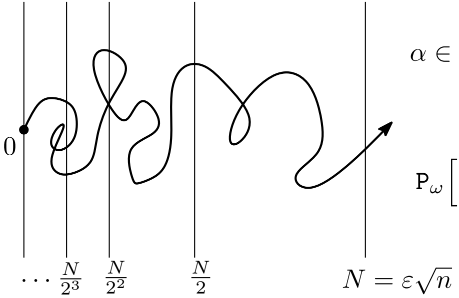

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observe that, for G_k, $\frac{\text{vertical size}}{\text{horizontal size}} \simeq (2\alpha)^k$

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- in particular, can one prove that $\frac{C_1}{n} \leq P_\omega[\text{cross the strip of width } n] \leq \frac{C_2}{n}$?