Optimal transport in non-commutative probability

① **Two views on heat flow:**

* Classical L^2-theory

- Gauss space $L^2(\mathbb{R}^n, \gamma)$. $\gamma(dx) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx$.
- Dirichlet energy $\mathcal{E}_\gamma(\psi) = \int_{\mathbb{R}^n} |\nabla \psi(x)|^2 d\gamma(x)$.

-> Gradi. Flow eq. for $\mathcal{E}_\gamma$ in $L^2(\mathbb{R}^n, \gamma)$ is the Ornstein-Uhlenbeck equation: $\partial_t \psi = -\mathcal{L}\psi$.

where $-\mathcal{L}\psi(x) = \Delta\psi(x) - \langle x, \nabla\psi(x)\rangle$.

* optimal transport.

- 2-Wasserstein space $(\mathcal{P}_2(\mathbb{R}^n), W_2)$
- Relative entropy $\mathrm{Ent}_\gamma(\mu) = \int \rho(x) \log \rho(x) d\gamma(x)$, $\rho = \frac{d\mu}{d\gamma}$

⤷ GF of Ent_γ w.r.t. W_2. is the same O-U equation

② . **Quantum Mechanical Perspective on $L^2(\mathbb{R}^n, \gamma)$.**

$\bigoplus_{m \geq 0} (\mathbb{R}^n)^{\otimes_{sym} m}$ — particles. $\triangleq \mathcal{F}^n_{sym}$, Symmetric Fock space.

- Wiener: Canonical Isometry: $I : L^2(\mathbb{R}^n, \gamma) \longrightarrow \mathcal{F}^n_{sym}$.

$I : H_{m_1}(\langle\cdot, e_1\rangle) \cdots H_{m_n}(\langle\cdot, e_n\rangle) \longmapsto P_{sym}(e_1^{\otimes m_1} \otimes \cdots \otimes e_n^{\otimes m_n})$

- $\mathcal{N} = I \circ \mathcal{L} \circ I^{-1}$ is the boson number operator.

$\mathcal{N}h = m h$. $\forall\, h \in (\mathbb{R}^n)^{\otimes_{sym} m}$

③ Fermionic Counterpart

- Anti-symmetric Fock space: $\mathcal{F}^n_{\text{asym}} = \bigoplus_{m=0}^{n} (\mathbb{R}^n)^{\wedge m}$
- Fermionic Number Operator: $\mathcal{N} h = m h, \quad \forall\, h \in (\mathbb{R}^n)^{\wedge m}$

Def. let $Q_1, \dots, Q_n$ be self-adjoint operators on a Hilbert space s.t.

$$\begin{cases} Q_i Q_j = -Q_j Q_i & \forall\, i \neq j \\ Q_i^2 = I & \forall\, i \end{cases}$$

let C be the algebra $\{Q_1, \dots, Q_n\}$. ~~called~~, called Clifford algebra.

Basis of C are $Q_{\underline{\alpha}} = Q_1^{\alpha_1} \cdots Q_n^{\alpha_n}$ $\quad \alpha_1, \dots, \alpha_n \in \{0,1\}$.

Fermionic Wiener Isometry:

$$I : C \longrightarrow \mathcal{F}^n_{\text{asym}}$$

$$Q_{\underline{\alpha}} \longmapsto e_1^{\wedge \alpha_1} \wedge \cdots \wedge e_n^{\wedge \alpha_n}$$

④ Analysis on the Clifford algebra

1. N.C integration: $\tau : C \longrightarrow \mathbb{C}$, by

$$\tau(Q_\alpha) = \begin{cases} 1 & \underline{\alpha} = \underline{0} \\ 0 & \text{otherwise.} \end{cases}$$

(τ is a positive linear functional satisfying $\tau(AB) = \tau(BA)$.)

Remark. τ induces a scalar product

$$\langle A, B \rangle_{L^2} = \tau(A^* B).$$

2. NC. Differentiation: define $\partial_j : \mathcal{C} \to \mathcal{C}$ by

$$\partial_j A = \frac{1}{i}\left(Q_j A - \Gamma(A) Q_j\right), \quad \text{where } \Gamma(Q_{\underline{\alpha}}) = (-1)^{|\underline{\alpha}|} Q_{\underline{\alpha}}.$$

3. Def: (Clifford Dirichlet Form, Gauss)

$$\mathcal{E} : \mathcal{C} \times \mathcal{C} \to \mathbb{C}.$$

$$\mathcal{E}(A,B) = \sum_{j=1}^{n} \langle \partial_j A, \partial_j B \rangle_{L^2}.$$

4. Prop. The associated operator is the number operator

i.e. $\mathcal{E}(A,B) = \langle \mathcal{L} A, B \rangle_{L^2}$, where $\mathcal{L} = I^{-1} \circ N \circ I$

5. General setup.

- $(\mathcal{A}, \tau)$ non-commutative prob space, i.e.
  - $\mathcal{A}$ is a $*$-algebra of operators on a Hilbert space (finite dim)
  - τ $\mathcal{A} \to \mathbb{C}$ trace (positive linear functional s.t. $\tau(AB) = \tau(BA)$)
- $\partial_j : \mathcal{A} \to \mathcal{A}_j$, $j = 1, \dots, n$, operator of the form

$$\partial_j A = Q_j r_j(A) - l_j(A) Q_j, \text{ where.}$$

$(\mathcal{A}_j, \tau_j)$ non-comm. prob. space, $Q_j \in \mathcal{A}_j$. $l_j, r_j : \mathcal{A} \to \mathcal{A}_j$. algebra hom.

⑥. Examples:

1). Clifford algebra

2). Continuous time Markov chains with transition rates $Q(x,y)$ on reversible inv. measure π.

$$\mathcal{E}(\psi) = \frac{1}{2} \sum_{x,y} (\psi(x) - \psi(y))^2 Q(x,y)\pi(x).$$

$$\mathcal{L}\psi(x) = \sum_y Q(x,y)(\psi(y) - \psi(x)).$$

3). Lindblad equations (open quantum system)

⑦. Recall: Benamou-Brenier formula for $\rho_0, \rho_1 \in \mathcal{P}(\mathbb{R}^n)$

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^n} |\nabla\psi_t(x)|^2 \, d\rho_t(x)\, dt \; : \; \text{s.t. } \partial_t\rho + \nabla\cdot(\rho\nabla\psi) = 0, \; \begin{cases} \rho_{t=0} = \rho_0 \\ \rho_{t=1} = \rho_1 \end{cases} \right\}.$$

Problem: How to define "$\rho * \partial_j\psi$" ?

In $\mathbb{R}^n$, heat equation as continuity equation.

$$\partial_t\rho = \Delta\rho \iff \begin{cases} \partial_t\rho + \nabla\cdot(\rho\nabla\psi) = 0. \\ \psi = -\log\rho. \end{cases}$$

use chain rule: $\nabla\rho = \rho\cdot\nabla\log\rho$

⑧. Non-commutative Chain Rule

We have: $\partial_j(AB) = \partial_j(A)\cdot B + A\cdot\partial_j(B)$. (Assume $l = r = id$).

Induction: $\partial_j(A^n) = \sum_{k=0}^{n-1} A^k \cdot \partial_j A \cdot A^{n-k-1}$

Set $\rho = A^n$: $\partial_j\rho = \sum_{k=0}^{n-1} \rho^{\frac{k}{n}} \partial_j \rho^{\frac{1}{n}} \rho^{\frac{n-k-1}{n}}$

Then $\partial_j \rho = \frac{1}{n} \sum_{k=0}^{n-1} \rho^{\frac{k}{n}} \dfrac{\partial_j(\rho^{\frac{1}{n}} - I)}{\frac{1}{n}} \rho^{1-\frac{k+1}{n}}$.

Let $n \to \infty$,

Then $\partial_j \rho = \int_0^1 \rho^x \cdot \partial_j \log \rho \cdot \rho^{1-x} \, dx$

⑨ **Def**. Set $\mathcal{P}(A) = \{\rho \in A : \rho \geq 0, \, Tr(\rho) = 1\}$. (prob. densities)

Def ("2-Wasserstein metric")

For $\rho_0, \rho_1 \in \mathcal{P}(A)$, set

$$W^2(\rho_0, \rho_1) = \inf \left\{ \int_0^1 \langle \rho_t * \partial_j \psi_t, \partial_j \psi_t \rangle_{L^2} \, dt . \; \middle| \; \begin{cases} \partial_t \rho + \sum \partial_j^*(\rho_t * \partial_j \psi_t) = 0. \\ \rho_{t=0} = \rho_0, \; \rho_{t=1} = \rho_1 \end{cases} \right\}$$

where $\rho * A = \int_0^1 l_j(\rho)^{1-x} A \, r_j(\rho)^x \, dx$.

⑩. **Prop**. W is associated to Riemannian ~~Structure~~ on $\mathcal{P}(A)$.

Thm. Set $\mathrm{Ent}(\rho) = \tau(\rho \log \rho)$ (von Neumann entropy)

The grad. flow equation. for Ent w.r.t W is

$$\partial_t \rho = -\mathcal{L} \rho, \quad \text{where } \mathcal{L} = \sum_{j=1}^n \partial_j^* \partial_j .$$

In eg. 1). (Carlen - M)

eg. 2). (Mielke, Chow-Huang-Li-Zhou, M.)

eg. 3). Mielke.

(11). Thm (Carlen - Maas)

If $(A, \tau, (\partial_j))$, satisfies a log-Sobolev. ineq.

i.e $\mathrm{Ent}(\rho) \leq \frac{1}{2K} \sum_j \rho\langle \partial_j \rho, \partial_j \log \rho \rangle$

Then non-comm. Talagrand ~~ie~~ ineq. holds.

$$\left(\text{i.e. } W(\rho, I)^2 \leq \frac{2}{K} \mathrm{Ent}(\rho) \quad \forall \rho\right).$$

(analogue of Otto-Villani).