Definition

A knot is a smooth embedding of the circle into 3-dimensional space.
Figure: A knot
Fact I: The mathematical study of knots includes several different branches, each with a very different "flavour": algebraic, geometric, ...
Fact I: The mathematical study of knots includes several different branches, each with a very different “flavour”: algebraic, geometric, ...

Fact II: Knots relate to nature, for instance via the tangles in long strands of DNA.
Definition

Given a knot $K$ in 3-dimensional space, an orientable (2-sided) surface $S$ that has boundary $K$ is called a *spanning surface*. 
Figure: A Seifert surface

Theorem

(Seifert) Every knot admits a spanning surface.
Seifert surfaces

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Seifert’s algorithm

- Step 1: Consider a projection of the knot and give it an orientation.
- Step 2: Resolve each crossing to obtain Seifert circles.
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- Step 3: The Seifert circles bound disks with a natural orientation.
Seifert surfaces

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Seifert’s algorithm

- Step 1: Consider a projection of the knot and give it an orientation.
- Step 2: Resolve each crossing to obtain Seifert circles.
- Step 3: The Seifert circles bound disks with a natural orientation.
- Step 4: Connect disks via bands to obtain a spanning surface.
For a surface $S$ constructed via Seifert’s algorithm, the Euler characteristic of $S$, denoted by $\chi(S)$, can be computed as

$$\chi(S) = \#\text{disks} - \#\text{bands}$$

**Definition**

A *Seifert surface* for a knot $K$ is a spanning surface of maximal Euler characteristic.
Fact I: Many knots $K$ in $\mathbb{S}^3$ admit non-isotopic Seifert surfaces. (Eisner 1977)
**Fact I:** Many knots $K$ in $S^3$ admit non-isotopic Seifert surfaces. (Eisner 1977)

**Fact II:** Many knots $K$ in $S^3$ admit disjoint non-isotopic Seifert surfaces. (Eisner 1977)
Figure: Connect sum of knots
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Seifert Surfaces

Figure: Schematic

Jennifer Schultens
Knots and surfaces in 3-dimensional space
Kakimizu complex

Definition

The vertices of the Kakimizu complex $\text{Kak}(K)$ of a knot $K$ in $\mathbb{S}^3$ are given by the isotopy classes of minimal genus Seifert surfaces for $K$. 
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The $n$-simplices of the Kakimizu complex of $K$, for $n > 1$, are given by $n$-tuples of vertices that admit representatives that are pairwise disjoint.
Examples of Kakimizu complexes

- **Example I:** Fibered knots have trivial Kakimizu complexes.
Examples of Kakimizu complexes

- **Example I:** Fibered knots have trivial Kakimizu complexes.
- **Example II:** Hyperbolic knots have finite Kakimizu complexes.

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Knots and surfaces in 3-dimensional space
Theorem

\textit{(Scharlemann-Thompson)} The Kakimizu complex of a knot is connected.

Not stated in these terms.
Theorem

(Kakimizu) Suppose that $K_1, K_2$ are knots with unique minimal genus Seifert surfaces (up to isotopy). Then the Kakimizu complex of $K = K_1 \# K_2$ is a bi-infinite ray.

More recently, Kakimizu computed the Kakimizu complexes for all prime knots with up to 10 crossings.
(Banks) Suppose that $K_1, K_2$ are knots, then the Kakimizu complex of $K_1 \# K_2$ is the product of three complexes: The Kakimizu complex of $K_1$, the Kakimizu complex of $K_2$ and the complex that has underlying space $\mathbb{R}$ and vertices at the integers.
Structure of Kakimizu complexes

Theorem

(Banks) There exist knots with locally infinite Kakimizu complex.

Theorem

(Banks) A knot has locally infinite Kakimizu complex only if it is a satellite of either a torus knot, a cable knot or a connected sum, with winding number 0.
Figure: An essential torus in a knot complement
(S 2007) The Kakimizu complex of a knot $K$ is a flag complex.
Structure of the Kakimizu complex

Theorem

(S 2007) The Kakimizu complex of a knot $K$ is a flag complex.

Theorem

(Kapovich 2009) Let $M$ be a Riemannian 3-manifold with smooth strictly convex boundary, together with a compact family $\mathcal{J}$ of smooth curves on $\partial M$. Let $f_i : (S_i, \partial S_i) \rightarrow (M, \mathcal{J})$, $i = 1, \ldots, n$ be incompressible surfaces which are pairwise non-isotopic and pairwise disjoint. Let $g_i : (S_i, \partial S_i) \rightarrow (M, \mathcal{J})$, $i = 1, \ldots, n$ be relative area minimizers in the proper isotopy classes of $f_i$, $i = 1, \ldots, n$. Then $g_1(S_1), \ldots, g_n(S_n)$ are also pairwise disjoint.
The Kakimizu complex of a knot is contractible.

(Przytycki-S 2010) The Kakimizu complex of a knot is contractible.
Implicit in Kakimizu’s work is a projection map (coming from considerations involving covering spaces and Kakimizu’s formulation of the distance on the Kakimizu complex) that, given two vertices $v, w$, produces a vertex $\pi_v(w)$ that is one step closer to $v$ than $w$.

\[ d(v, \pi_v(w)) = d(v, w) - 1 \]
Theorem

(Przytycki-S 2010) The Kakimizu complex of a knot is contractible.

Idea of proof: Choose a vertex $v$ in $Kak(K)$ and prove that the projection map onto $v$ is a contraction of $Kak(K)$.

Challenge: Make sure that the projection map behaves well on links of vertices.
Theorem

(Scharlemann-Thompson) The Kakimizu complex of a knot is connected.

Idea of new proof: Given any two vertices $v, w$, we construct a path