ALGEBRAIC GEOMETRY OF TOPOLOGICAL FIELD THEORIES

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This was a chalk talk. The speaker's notes may be found at the bottom.

1. Introductory Remarks

This work is joint with D. Nadler, A. Neitzke, and T. Nevins. It's part of an FRG group with Freed, Frenkel, Hopkins, Moore, and Telemon.

Goal: explore mysterious object coming out of physics called Theory $\mathcal{X}$. Physicists call this the six dimensional (2,0) SCFT.

This involves going from physics to representation theory and the topology comes in as part of the dictionary. There is still much work to do related to the foundations. What we're presenting here is what should be true (based on physical intuition) and consequences once Theory $\mathcal{X}$ with the prescribed properties has been constructed.

There will be a workshop on related material at Banff called BIRS May 24-29, 2015.

2. Rough features of theory $\mathcal{X}$

Everywhere we are working over $\mathbb{C}$.

Big picture: begin with a simply laced Lie algebra $\mathfrak{g}$ and get a 6-dimensional conformal field theory.

The accessible part of this is a 2-dimensional conformal field theory (CFT) valued in a 4-dimensional topological field theory (TFT).

Given a Riemann surface Σ we associate $\mathcal{X}_\Sigma$, an oriented 4-dimensional TFT. In a recent paper Freed and Teleman describe the structure of this object. We will related these objects to representation theory.

We model $\mathcal{X}_\Sigma$ using B-models (or 'B type Σ' models). A B-model is provided by looking at maps from simplicial sets into $\mathcal{M}$, where $\mathcal{M}$ is a scheme or stack.

For example if $\mathcal{M} = \operatorname{Spec} R$ then $\mathcal{O}(\mathcal{M}^X) = \mathcal{O}(\mathcal{M}) \otimes X$ and $\mathcal{O}(\mathcal{M}^{S^1}) = \mathcal{O}(\mathcal{M}) \otimes S^1$. This is the DAG-style mapping space.

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In the derived category $QC(\mathcal{M}^X) = QC(\mathcal{M}) \otimes \Sigma$ where $\mathcal{M}$ is a nice object (e.g. a scheme or stack), and $QC(\mathcal{M})$ is quasi-coherent sheaves over $\mathcal{M}$. This example has been worked out by Ben-Zvi, Freed, Nadler.

B-model is a 2d TFT where to a point we associate $QC(\mathcal{M})$ thought of as a dg-category.

Rowanski-Witten theory (RW-theory) is a 3d TFT where to a point we associate $QC(\mathcal{M})$ thought of as a monoidal category. This is the 3d TFT attached to a holomorphic symplectic manifold.

Formally, $RW_{T^*M}(S^1) = QC(\mathcal{M}) \otimes S^1 = QC(\mathscr{L}M)$ where $\mathscr{L}\mathcal{M}$ is the loop space. By HKR this is equal to $QC(T_{\mathcal{M}}[-1])$. Koszul duality (if you complete along $\mathcal{M}$) gives an isomorphism to $QC(T^*\mathcal{M})$. This justifies the definition above.

3. Moduli spaces and 4d TFTs

Let Z be a 4d TFT (we'll say a word about what we mean at the end of the section). It's taking a 3-manifold to a chain complex and a 2-manifold to a dg category. The motivation is from super-symmetric gauge theory. This is the type of data the physics is providing.

We want to define the moduli space of Z. We will need to consider so-called 'local operators' in Z. Morally, local operators are things you can insert points into.

We define the *chain complex of local operators* to be $Z(S^3)$. It has the natural structure of an E_4-algebra. The motivation is by looking at a 4-manifold as a cobordism between 3-manifolds, e.g. from $M^3 \coprod S^3$ to M^3. Then for any point $x \in M$ one can then look at what happens when you cut out a ball around x and carry that ball through the cobordism. That provides an action of $Z(M^3)$ on $Z(S^3)$ and so $Z(S^3)$ is a $Z(M^3)$-module.

The factorization homology is $Z(M) \in \int_M Z(S^3)$. Think of $\int_M Z(S^3)$ as $Z(S^3) \otimes M$ in a similar way to the $- \otimes -$ in the previous section but now for E_4-algebras rather than commutative algebras.

Define the moduli space $\mathcal{M}_4$ as $Spec(Z(S^3))$. This is an E_4-algebra, i.e. a commutative ring together with an odd Poisson bracket.

Basically we are trying to do algebraic geometry where we replace commutative rings by E_4-algebras.

By construction, $Z(S^3)$ corresponds to functions on $\mathcal{M}_4$, i.e. what physicists would call vev's or vacuum expectation values. So $Z(M^3)$ is a sheaf on $\mathcal{M}_4$.

Now that we have the moduli space $\mathcal{M}_4$ in hand we can construct the $\mathcal{X}$ promised at the start.

The 4d TFT's we're considering have a small subtlety. We're really asking for a 4d TFT with 4-dualizability *relative to* $Z(S^3)$. In particular, $Z(S^3 \times S^1)$ is not well-defined. Think of this "relative to $Z(S^3)$" condition as a finiteness condition in the background, analogous to the following situation in algebra. Any algebra has a center, which can be finite dimensional. S^3 is like the center of a 3d TFT, and we're working with $Z(S^3)$ so this finiteness condition has to be built into what we mean by a 4d TFT. See the speaker's written work for more details.

4. First attempt at understanding $\mathcal{X}$

Let Σ be a Riemann surface. Then $\mathcal{X}_\Sigma$ is supposed to be a 4d TFT. This will be done via the M_4 from above. Define M_4^{even} to be the Hitchin base $\bigoplus H^0(\Sigma, \Omega^{\otimes d_i})$. The $\mathbb{C}^*$ action on $\Omega^{\otimes d_i}$ gives a cohomological grading. Observe that there is no interesting E_4-structure, because of the grading.

Let Z_{S^1} be a 3d TFT given by $Z_{S^1}(M) = Z(S^1 \times M)$. This gets you from the 4d TFT Z to a 3d TFT. We can play this trick again:

Follow the previous section and construct a 3d moduli space $\overline{\mathcal{M}_3}$ as Spec of local operators in Z_{S^1} and then define $Z_{S^1}(S^2) = Z(S^1 \times S^2)$ as an E_3-algebra. Can then form even Poisson algebra as above. So now there's an affine Poisson structure of degree 2.

In our examples this is uninteresting because for $Z = \mathcal{X}_\Sigma$, the resulting $\overline{\mathcal{M}_3}$ is just $\mathcal{M}_4$. So we must have done something wrong.

5. Second attempt

We can turn to the physics to see what we did wrong. Physicists look at the 3d moduli space of a 4d gauge theory as the total space of Seiberg-Witten theory, formed from integral systems. In this light, we should not have tried to built our moduli space from affine data.

So we go back and look at $Z(-)$ as:

$Z(S^2)$ = line defects

$Z(S^1)$ = surface defects

$Z(S^0)$ = surface domain walls

Line defects are defined on 1-manifolds embedded in 4-manifolds by way of Lurie's tangle hypothesis. The other two cases are similar.

Consider line operators on $Z(S^2)$. This is now an E_3-category with the same operations as before on vector spaces, labeled by 3-disks. The unit sitting in the E_3-category is just $Z(D^3)$, i.e. you fill in one of the S^2's sitting in M. Now,

$\langle 1\rangle = End(1)$-modules $= Z(S^3)-$modules. And $Z(S^3)$ is an E_4-algebra, so we have strictly more structure than in our first attempt.

As in the previous section you can go down a layer and get an E_2-category, etc. In physics these line operators are known as Wilson-Hooft operators.

Use line operators in Z_{S^1}, i.e. $Z_{S^1}(S^1) = Z(T^2)$ for the torus T^2. This is now an E_2-braided category.

Recall that our goal is to get an interesting 3d TFT. If we were working with a symmetric monoidal category then we'd use Tannakian formalism at this point. In fact this will work, as we now sketch.

The idea of Tannakian formalism is that if C is a symmetric monoidal ∞-category then you can try to realize it as $QC(\mathcal{M})$ for some scheme or stack $\mathcal{M}$. So now we should try to realize our braided category as $QC(\mathcal{M})\otimes$ (something). For example, if C is $Rep(G)$ then $\mathcal{M} = BG$. In general one finds $\mathcal{M}$ by Yoneda embedding.

Let R be a commutative ring. Define Spec $C(R) = Hom_{\otimes}(C, R\text{-mod})$. This object wants to be an algebraic stack, and if C is given by the method above then it is one.

DAGVIII and Wallbridge explain well the connection between C and $QC(\operatorname{Spec} C)$.

Note: we have been ignoring connectivity. Would need t-structures to do formally, and this has been done.

Let C be an E_n-category. Test it against R-mod as above and you get an RE_{n+1}-algebra. So $\operatorname{Spec} C$ is an E_{n+1}-stack and in the best case scenario it's actually $QC(\operatorname{Spec} C)$. This is not true for a random braided tensor category but we do expect this to hold for Theory $\mathcal{X}$. So we will assume this and leave it to the topologists to prove that it works.

We are looking at $Z_{S^1}(S^1)$ as an E_2-category. We get $\mathcal{M}_3 = \operatorname{Spec} Z_{S^1}(S^1)$ as an E_3-stack (in particular, an even Poisson stack).

Then we define $\mathcal{X}_\Sigma$ to be the association $\mathcal{M}_3 \leftrightarrow T^*Bun_O(\Sigma)$ (a.k.a. Hitchin space coming from the stack Bun_O), making use of the action of $\mathbb{C}^*$ to get the appropriate grading.

A physicist would say that we're approximating the compactification of Z on S^1 by a RW-theory on the moduli space $RW_{\mathcal{M}_3}$. See the work of Garotto-Moore-Neitzke.

Given a scheme Y, we have said $QC(T^*Y)$ looks like $QC(\mathscr{L}Y)$ (at least, locally and after doing a completion) which is an E_2-category.

6. Using this object we've defined

6.1. **Structures on $Z(T^2)$ and on $\mathcal{M}_3$.** Because this is an E_2-category there is a unit object $O_{\mathcal{M}_3} = Z(S^1 \times D^2) \in Z(S^1 \times S^1)$, given by the structure sheaf $\Gamma(\mathcal{M}_3, O) = Hom(O, O) = Z(S^1 \times S^2)$ and $O(\overline{\mathcal{M}_3}) = Z(S^1 \times S^2)$.

Choice of a point $x \in T^2$ (the 2-torus) gives a map $\phi : \mathcal{M}_3 \to \mathcal{M}_4$, the moduli space we started with. This is a map of E_3-schemes where the codomain is given the trivial E_3-structure (e.g. $\mathcal{M}_3 \to [T^2, \mathcal{M}_4] \to \mathcal{M}_4$). More structure can be obtained by considering the factorization homology. This ϕ is a collection of Poisson commuting functions, which is starting to look like an integral system, so we are getting closer to the physics.

One can also define a dual integral system. Previously we worked with $Z(S^1 \times S^1)$ and got the braided monoidal structure by looking at the second S^1. If we instead look at the first S^1 then we get the dual integral system. We have an $SL_2\mathbb{Z}$ worth of E_2-structures. We also get

$$\begin{tikzcd}
Z(T^2) = \mathcal{M}_3 \arrow[rr, dotted, leftrightarrow] \arrow[dr] & & \mathcal{M}_3^\vee = \mathrm{Spec}(Z(S^1 \times S^1)) \arrow[dl] \\
& \mathcal{M}_4 &
\end{tikzcd}$$

As stacks $\mathcal{M}_3$ and $\mathcal{M}_3^\vee$ are identical, but there are non-trivial derived self-equivalences.

If our conjecture regarding Theory $\mathcal{X}$ is accurate (after Theory $\mathcal{X}$ is constructed that is) then this will correspond to the following structure in the physics world:

$$\begin{tikzcd}
T^*Bun_G^\Sigma \arrow[rr, dotted, leftrightarrow] \arrow[dr] & & T^*Bun_{G^\vee}^\Sigma \arrow[dl] \\
& Hitch &
\end{tikzcd}$$

6.2. **Hitchin Section.** We want a section of the map $\mathcal{M}_3 \to \mathcal{M}_4$. That's like saying $\mathcal{M}_3$ has two monoidal structures and the second one is convolution.

$O^\vee = O_{\mathcal{M}_3^\vee} = Z(D^2 \times S^1) \in QC(\mathcal{M}_3)$ is the unit for the convolution structure on $QC(\mathcal{M}_3)$.

$\Gamma(\mathcal{M}_3, O^\vee) = Hom(O, O^\vee) = Z(D^2 \times S^1 \coprod_{T^2} S^1 \times D^2) = Z(S^3) = O(\mathcal{M}_4)$

The compatibility of $\otimes$ and $*$ gives a map $\mathcal{M}_3 \to \mathcal{M}_4$, i.e. a family of abelian groups which is compatible from the eyes of $\mathcal{M}_4$. That's really what an integrable system is, and this is our punchline. The 4d TFT knows about the integrable system. See the work of Ngô.

6.3. **Quantization of integrable systems.** This is based on the work of on Ω-deformations.

There is a tautological way to quantize $Z(S^1 \times S^1)$ by passing to S^1-equivariant family $Z(S^1 \times S^1)^{S^1}$. This is a family over $C^*(BS^1) = \mathbb{C}[\epsilon]$ where $|\epsilon| = 2$.

Deformation quantization of $\mathcal{M}_3$ gives an association $QC(\mathcal{M}_3) \to QC_\epsilon(\mathcal{M}_3) = Z(S^1 \times S^1)^{S^1}$

Under our conjecture this corresponds to $QC(T^*Bun_G) \to \mathscr{D}(Bun_G)$ where $\mathscr{D}$ is for D-modules. So this is saying in particular that $QC(\mathscr{L}X)^{S^1} \leftrightarrow \mathscr{D}$-mod$(X)$, i.e. X knows how to quantize itself.

We get a similar picture as from the beginning of this section, but with ϵ:

$$\begin{tikzcd}
\mathcal{M}_{3,\epsilon} \arrow[rr, dotted, leftrightarrow] \arrow[dr] & & \mathcal{M}^v_{3,\epsilon} \arrow[dl] \\
& \mathcal{M}_4 &
\end{tikzcd}$$

This material also relates to Geometric Langlands, which provides

$\mathscr{D}$-mod$(Bun_G\Sigma) \simeq QC(Loc_{G_v}\Sigma)$

One could also do deformation quantization on both S^1's and get a more complicated version of Geometric Langlands.

There are many similar games you can play.

Algebraic Geometry of Topological Field Theories

report work in progress w/ Nadler Neitzke Nevins – part of FRG with

Freed Frenkel Hopkins Moore & Teleman. Goal: explore 6d CFT

(6d (2,0) SCFT), formulate structures mathematically, make predictions for geometric rep theory...

BIRS May 24-29 2015!

"Reimagining foundations" – will imagine these are foundations for what follows!

Rough features: ADE Dynkin diagram $\longmapsto$ 6d $\mathfrak{g}$FT

really relative to 7d TFT, Freed-Teleman – need to choose extra structure on 6-manifolds,

roughly Lagrangian in $H^3(X,\Lambda)$ ... pick out groups, etc

Most accessible part: twisted version, a 2d CFT valued in 4d TFTs:

Σ Riemann surface $\rightsquigarrow \mathcal{X}_\Sigma$ 4d extended "dg" topological field theory $\left(\begin{smallmatrix}3\text{-manifold} \rightsquigarrow \text{complex} \\ 2\text{-manifold} \rightsquigarrow \text{dg cat} \\ \dots\end{smallmatrix}\right)$

How to understand structure of $\mathcal{X}_\Sigma$?

Toy models for TFTs from algebraic geometry: B-type σ-models.

Fix a scheme or stack $\mathcal{M}$, study maps of manifolds (as s.sets / homotopy types)

into $\mathcal{M}$: eg if $\mathcal{M}=\operatorname{Spec} R$, $\mathcal{O}(\mathcal{M}^\Sigma) = \mathcal{O}(\mathcal{M}) \otimes \Sigma$ derived scheme $(/\mathbb{C})$
cdg, i.e. E_∞-algebra

or $\mathcal{M}$ nice (eg perfect) stack, $QC(\mathcal{M}^\Sigma) = QC(\mathcal{M}) \otimes \Sigma$ ...

makes sense in any dimension, define "B-model" TFTs (special case of factorization homology)

eg $\bullet \longmapsto QC(\mathcal{M})$ as monoidal category gets 3d TFT (or rather its 0,1,2 part):

Rozansky-Witten theory $RW_{T^*\mathcal{M}}$ [hol symplectic, completed along $\mathcal{M}$] – BFN, KRS,...

Why T^*M? $RW_{T^*M}(S^1) = \overbrace{QC(M)\otimes S^1}^{\mathcal{L}/H_*} = QC(\mathcal{L}M) \overset{HKR}{=} QC(T_M[-1])$

$\overset{\text{Koszul}}{\longleftrightarrow} QC(T^*M)$ completed along M

ie 2d B-model of T^*M is obtained by compactifying $\underline{RW_{T^*M}}$ on S^1

$Z_{S^1}(X) = Z(S^1 \times X)$.

Z 4d TFT : try to model by such B-models. Physics: describe moduli space of theory

Start with local operators: dg vector space $Z(S^3)$. Structure (OPE):

E_4 algebra — by formality can think of as commutative + odd (deg 3) Poisson str

Acts on any $Z(M^3)$: or anything else once we choose a point —

.... $Z(M) \in \int_M Z(S^3)$ factorization homology — "$Z(S^3)\otimes M$" for E_n algebras (M n-manifold)

[or coaffine stack]

Think of this geometrically: $\mathcal{M}_4 := \operatorname{Spec} Z(S^3)$ affine E_4 scheme, roughly: odd Poisson derived scheme

moduli of Z: local operators $O \in Z(S^3)$ give functions (vev's) on $\mathcal{M}_4$,

$Z(M^3)$ gives qc sheaf on $\mathcal{M}_4$, etc!

[Note: If Z full 4d TFT, $Z(S^3)$ finite dim $\rightsquigarrow$ $\mathcal{M}$ = bunch of points...

Not what physicists mean by 4d TFT! better approximation:

$3+\varepsilon$ dim TFT (Walker) .... ie Morse index 0 cobordisms defined,

& fully dualizable relative to $Z(S^3)$ — ie relative to moduli space

Analogue: any assoc algebra has a center ... & much easier to be fin dim separable as $Z(A)$-module than over k!]

$\mathcal{X}_\Sigma$: $\mathcal{M}_4^{even}$ = Hitchin base $\oplus H^0(\Sigma, \Omega^{\otimes d_i}) = \Gamma(\Sigma, \mathfrak{h}^*/W \times_{\mathbb{C}^*} \Omega^\cdot)$ $\circlearrowleft \mathbb{C}^*$

as graded space - [Compare Spec $H^*(BG) = \mathfrak{h}^*/W$ (appears for special case - Donaldson theory)]

[ie work 2-periodically — or better, cohomological grading comes from

$\mathbb{C}^*$-action on entire theory - "R-symmetry", have shears of grading, identify

cohomological & geometric gradings]

Compactify to 3 dimensions: $Z_{S^1}(M) = Z(S^1 \times M)$

$\rightsquigarrow \overline{\mathcal{M}_3}$ = Spec of local operators in 3d = Spec $Z(S^1 \times S^2)$, E_3 = even Poisson algebra

[even] Poisson variety

In some examples get nice symplectic affine varieties...

but in our case $Z = \mathcal{X}_\Sigma$ boring : $\overline{\mathcal{M}_3} \xrightarrow{\sim} \mathcal{M}_4$ just Hitchin base w/ $\{\,\} = 0$

Physics suggests 3d moduli space is total space of Seiberg-Witten integrable system

of $N=2$ gauge theory - won't detect an abelian variety fibration by its functions!

Higher codimension defects & spheres

$Z(S^2)$ = line defects
$Z(S^1)$ = surface defects
[$Z(S^0)$ = (self) domain walls]

limits of line / surface / hypersurface

Lurie tangle hypothesis: dualizable objects define 1/2/3 dim TFTs "coupled to bulk" ... defined on embedded submanifolds

$Z(S^2)$ E^3 category $\supset Z(S^3)$-mod $= \langle 1 \rangle_{Z(S^2)}$ (TFT many of suspension...)

- refines information given by local operators... Hecke functors ('t Hooft/Wilson loops)

[$Z(S^2) \longrightarrow End_{\circlearrowleft} Z(M)$
$\quad 1 \circlearrowleft_{Z(S^3)} \longmapsto Id \circlearrowleft_{Z(S^3)}$

- get more accurate description $Z(M) \in \int_M Z(S^2)$]

We can use line operators rather than local operators to probe the theory!

For Z_{S^1}: line operators form the E_2 category $Z_{S^1}(S^1) = Z(T^2)$

Tannakian formalism: given $\mathcal{C}$ symmetric monoidal category try to realize

$\mathcal{C} = QC(X)$ for a stack X. eg $\mathcal{C} = \mathrm{Rep}\, G \rightsquigarrow X = BG$.

Tautological (Yoneda) guess for X [right adjoint to functor QC: Stack $\to \otimes\text{-cat}^{op}$]

test $\mathcal{C}$ against R-mod, R commutative: $\mathrm{Spec}\,\mathcal{C}(R) = \mathrm{Hom}_{\otimes}(\mathcal{C}, R\text{-mod})$ [should keep track of connectivity - t-structure]

- works very often in geometric settings (DAG VIII, Wallbridge, ..)

$\mathcal{C}$ E_n category - "I" imagine analogous theory works [learned from Francis]

R E_{n+1} algebra $\rightsquigarrow$ R-mod E_n-category, write same definition $\rightsquigarrow$ E_{n+1} stack.

Our case: E_2 category $Z_{S^1}(S^1) \rightsquigarrow E_3$ (even Poisson) stack $\mathcal{M}_3 := \mathrm{Spec}\, Z_{S^1}(S^1)$

Expectation
- Tannakian reconstruction works here: $Z(S^1 \times S^1) = QC(\mathcal{M}_3)$
- $\mathcal{M}_3 \sim T^*\mathrm{Bun}_G \Sigma$ Hitchin space (moduli of Higgs bundles)

[or rather Ngô approximation thereof] [N.B. - t structures]

- NB $T^*\mathrm{Bun}_G \Sigma \circlearrowleft \mathbb{C}^*$ "R-symmetry" rescaling symplectic form -

Poisson bracket has degree -2 like E_3 algebra ... [should be ordinary stack with 2-shifted symplectic structure as in PTVV]

Physics: approximate Z_{S^1} by $RW_{\mathcal{M}_3}$ (cf GMN)

Note $QC(T^*Y) \overset{\text{Koszul}}{\simeq} QC(\mathcal{L}Y)$ naturally E_2 from $\{8 \to Y\}$

- similar E_2 structure for algebraic integrable systems

Some structures :

- Object $\mathcal{O}_{\mathcal{M}_3} = Z(S^1\times D^2) \in Z(S^1\times S^1)$ structure sheaf (monoidal unit)

  $\Gamma(\mathcal{M}_3, \mathcal{O}) = \mathrm{End}\,\mathcal{O}_{\mathcal{M}_3} = Z(S^1\times S^2) = \mathcal{O}(\overline{\mathcal{M}_3})$ affinization

SW ("integrable system")

- Choice of $x\in T^2$ gives E_3-map ${}'\mathcal{M}_3 \longrightarrow (\mathcal{M}_4, \{,\}=0)$ Hitchin integrable system $T^*\mathrm{Bun}_G\Sigma \longrightarrow \mathrm{Hitch}\,\Sigma$

  ie Poisson commuting Hamiltonians! comes from central action of $Z(S^3)$ on everything

  $Z(S^3) \to \mathrm{End}\,\mathcal{O}_{\mathcal{M}_3}$. [Really have $Z(T^2) \in \int_{T^2} Z(S^3)$

  $\leftrightarrow$ E_3 morphism

$$\begin{tikzcd} {}'\mathcal{M}_3 \arrow[r] \arrow[dr] & {[T^2, \mathcal{M}_4]} \arrow[d, "\text{choice of } x"] \\ & \mathcal{M}_4 \end{tikzcd}$$

]

- Dual integrable system: $Z(\underline{S^1}\times S^1)$ & $Z(S^1\times \underline{S^1})$ two E_2 structures...

$$\begin{tikzcd} \mathcal{M}_3 \arrow[dr] & & \mathcal{M}_3^\vee \arrow[dl] \\ & \mathcal{M}_4 & \end{tikzcd}$$

dual integrable systems

$$\begin{tikzcd} T^*\mathrm{Bun}_G\,\Sigma \arrow[dr] & & T^*\mathrm{Bun}_{G^\vee}\,\Sigma \arrow[dl] \\ & \mathrm{Hitch} & \end{tikzcd}$$

+ derived equivalence ("mirror symmetry") - part of $SL_2\mathbb{Z}$ symmetry of $Z(T^2)$

[symmetry broken by relative theory structure of $\mathcal{X}_\Sigma$ ..]

- Hitchin section : $\mathcal{O}^\vee = Z(D^2\times S^1)$ is unit of convolution structure on $QC(\mathcal{M}_3)$

  – defines section of ${}'\mathcal{M}_3 \to \mathcal{M}_4$:

$$\Gamma(\mathcal{M}_3, \mathcal{O}^\vee) = \mathrm{Hom}(\mathcal{O}, \mathcal{O}^\vee) = Z\Big(D^2\times S^1 \coprod_{T^2} S^1\times D^2\Big) = Z(S^3) = \mathcal{O}(\mathcal{M}_4)$$

- Compatibility of $\otimes, *$ $\Longrightarrow$ $\mathcal{M}_3 \longrightarrow \mathcal{M}_4$ is a sheaf of abelian groups

  – Ngô group scheme integrating Hitchin system!

Note only get abelian group relative to $\mathcal{M}_4$ since our theory is only 4-dualizable relative to $\mathcal{M}_4$: need (pair of pants) × (pair of pants) ....

• Quantization of integrable system: take S^1-equivariant objects (Nekrasov–Shatashvili; Ω-deformation)

$Z(S^1 \times \underline{S}^1)^{S^1}$ – linear over $\mathrm{Spec}(C^*(BS^1) = \mathbb{C}[\varepsilon]_{|\varepsilon|=2})$ – degree exactly lines up with degree of Poisson structure, get deformation quantization of $\widetilde{\mathcal{M}}_3$ (eg $QC(T^*Y)^{S^1} \longleftrightarrow \mathcal{D}(Y)$ modules) BZ-Nadler)

Note $Z(\underset{\varepsilon_2}{\underline{S}^1} \times S^1)^{S^1}$ still E_2 – ie dual E_3 stack $\mathcal{M}_3^\vee$ just gets deformed

$\mathcal{X}_\varepsilon$: $\mathcal{D}(\mathrm{Bun}_G \Sigma) \longleftrightarrow QC(\mathrm{Loc}_{G^\vee} \Sigma)$.

geometric Langlands correspondence

$Z(S^3)_\varepsilon \to Z(S^1 \times S^2_\varepsilon)$ recover Beilinson–Drinfeld quantization of Hitchin hamiltonians

• Turn on other ε: $Z(T^2)^{S^1 \times S^1}$ family over $\varepsilon_1, \varepsilon_2$: now lost both monoidal structures. $\mathcal{X}_\varepsilon$: $\mathcal{D}_k(\mathrm{Bun}_G \Sigma) \longleftrightarrow \mathcal{D}_{k^\vee}(\mathrm{Bun}_{G^\vee} \Sigma)$ quantum geometric Langlands

• $S^3 \rightsquigarrow \mathcal{X}_\Sigma(S^3_{\varepsilon_1,\varepsilon_2}) = \int_\Sigma \mathcal{X}_{\text{annulus}}(S^3_{\varepsilon_1,\varepsilon_2})$ chiral homology / conformal blocks of "holomorphic E_2 algebra" = vertex algebra: the $\mathcal{W}$-algebra of $\mathfrak{g}$

—(form of) AGT conjecture

• Other applications: $Z(T^2)_{\varepsilon_1,\varepsilon_2} \supset$ full subcategories equivalent to modules for Cherednik-type Hecke algebras $Z(S^1 \times \odot)_{\varepsilon_1,\varepsilon_2}$ labeled by surface defect $\in Z(S^1)$

...