CO-SEGAL ALGEBRAS AND DELIGNES CONJECTURE

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This was a chalk talk. The speaker decided not to share his hand-written lecture notes.

This is following earlier work of Kock-Toën.

1. Background

In the classical setting you fix a commutative ring K. For simplicity let's think of it as a field. Let A be a K-algebra with multiplication $\mu : A \otimes_K A \to A$.

Define the Hochschild complex with coefficients in A $C^0(A, A) \to C^1(A, A) \to \cdots \to C^h(A, A)$ where each $C^n(A, A) = \mathrm{Hom}_K(A^{\otimes n}, A)$. This is the n^{th} space of the endomorphism operad End_A, so it's containing the information of the algebraic structure of A.

Define the Hochschild homology $HH(A)$ to be the cohomology of this complex. This depends on μ.

Remark: there is a complex BA called the bar complex of A

$$[\cdots \to A^{\otimes n} = A \otimes_K A \otimes_K \cdots \otimes_K A \to \cdots \to A \otimes A \otimes A \to A \otimes A] \xrightarrow{\mu} A$$

So $BA_i = A \otimes A^{\otimes i} \otimes A = F(A^{\otimes i})$ where $F : K\text{-mod} \to \mathrm{Bimod}_A$ is left adjoint to the forgetful functor U. This adjunction gives $\mathrm{Hom}_{K-mod}(N, A) \simeq \mathrm{Hom}_{BiMod_A}(F(N), A)$.

Taking $N = A^{\otimes i}$ yields $\mathrm{Hom}_K(A^{\otimes i}, A) = \mathrm{Hom}_{BiMod}(F(A^{\otimes i}, A)$. In this light, the Hochschild complex is a Hom complex $\underline{\mathrm{Hom}}(BA, A)$ between two chain complexes of A-bimodules where we view A concentrated in degree 0.

If A is free or projective over K then BA is a resolution of A by bimodules, so $HH(A) = \underline{RHom}(A, A) = Ext(A, A)$.

2. Deligne's Conjecture

Deligne was hoping $HH(A)$ was something like an algebra over the 2-disk operad, i.e. if you draw a disk with two disks inside then this acts by taking $HH(A) \times HH(A) \to HH(A)$, where each of the two disks corresponds to a different multiplicative structure on the corresponding $HH(A)$. So we need to make sense of these different algebra structures on $HH(A)$.

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Theorem 2.1 (Kock-Toën)**.** *Suppose A is a simplicial algebra. Then the simplicial derived Hom space $REnd(A)$ is a simplicial 2-monoid.*

A simplicial 2-monoid has two compatible algebraic structures.

The non-derived version of the theorem looks at a monoidal category $(\mathcal{M}, \otimes, I)$ and outputs $(BiMod_A, \otimes_A, A)$. Under this assignment the right derived functor of $\mathrm{Hom}(I, I)$ is taken to $HH(A)$.

$\mathrm{Hom}(I, I)$ has two multiplicative structures:

(1) Given by composition.

(2) Given by $\mathrm{Hom}(I, I) \otimes \mathrm{Hom}(I, I) \to \mathrm{Hom}(I^2, I^2) \cong \mathrm{Hom}(I^2, I^2)$

A classical result of Eckmann-Hilton says that when you have two multiplications which are compatible then they provide a commutative structure on $\mathrm{Hom}(I, I)$.

Let $\mathcal{M}$ be a symmetric monoidal model category (more generally, a monoidal model category where $\mathrm{Hom}_\ell \simeq \mathrm{Hom}_r$ in the notation of Hovey's book). Then one can compute $REnd(I) = \underline{\mathrm{Hom}}(QI, RI)$ where QI is the cofibrant replacement of I and RI is the fibrant replacement of I.

We think of $REnd(I)$ as the Hochschild cohomology. These model category theoretic considerations provide $REnd(I) \simeq \underline{\mathrm{Hom}}(E, E)$ and this picks out the canonical multiplication $\mathrm{Hom}(E, E) \otimes \mathrm{Hom}(E, E) \to \mathrm{Hom}(E, E)$.

Now consider the multiplicative structure where you take two endomorphisms f and g to $f \otimes g$. In order for this to give a multiplication, we need a way to get from $\mathrm{Hom}(E^2, E^2)$ to $\mathrm{Hom}(E, E)$. The way to do this is via a zig-zag $\mathrm{Hom}(E^2, E^2) \to \mathrm{Hom}(E^2, E) \xleftarrow{\simeq} \mathrm{Hom}(E, E)$. So the multiplication is given by

$\mathrm{Hom}(E, E) \otimes \mathrm{Hom}(E, E) \to \mathrm{Hom}(E^2, E) \xleftarrow{\simeq} \mathrm{Hom}(E, E)$
i.e. $X(1) \otimes X(1) \to X(2) \leftarrow X(1)$

This is precisely the data of a coSegal algebra.

3. Co-Segal algebras

Let $\mathcal{M}$ be a monoidal category with a subcategory $\mathscr{W}$ of weak equivalences. A *co-Segal algebra X* is a lax-monoidal functor $X : (\Delta^+_{epi}, +, 0)^{op} \to (\mathcal{M}, \otimes, I)$ such that the underlying functor $X : (\Delta^+_{epi})^{op} \to \mathcal{M}$ factors through the subcategory of weak equivalences. This condition is the *co-Segal condition*.

Pictorially, we are requiring the following to be a homotopically constant diagram

$$\begin{array}{ccc} & & X(1) \\ & & \downarrow \\ X(1)\otimes X(1) & \longrightarrow & X(2) \\ \downarrow{\scriptstyle s\otimes 1} & & \downarrow\downarrow \\ X(2)\otimes X(1) & \longrightarrow & X(3) \end{array}$$

A Segal algebra is the data $X(1)\otimes X(1) \overset{\leftarrow}{\simeq} X(2) \to X(1)$.

Co-Segal algebras are very useful. They are in the background any time you have $S \otimes S \to S$ and $f : R \simeq S$. In particular, you have $R\otimes R \to S\otimes S \to S \leftarrow R$.

You also see co-Segal algebras in loop spaces, and it shows you $\Omega_*(X)$ is a co-Segal algebra with one object.

Let B be a dga. If the cohomology $H^*(B)$ is free then any cycle choosing map is a quasi-isomorphism $H^*(B) \to B$, and this makes the data $(B, H^*(B))$ into a co-Segal algebra.

Co-Segal algebra structure helps with the following problem. Given an operad $\emptyset$, when can you lift $\emptyset$-algebra structure to some B sitting over A, i.e. when does $\emptyset(n)\otimes A^{\otimes n} \to A$ lift along a map $B \to A$. This works if you take a cofibrant replacement $\emptyset_\infty$ of $\emptyset$.

4. Main Results

Theorem 4.1. *Let $\mathcal{M}$ be a symmetric monoidal model category. If $(\mathcal{V}, \otimes, U)$ is a symmetric monoidal, combinatorial model category satisfying the monoid axiom then there is a nice model structure on co-Segal algebras.*

This is constructed as a left Bousfield localization of the projective model structure on the diagram category (the one that appears in the definition of co-Segal algebra), where you precisely force the fibrant objects to be those satisfying the co-Segal condition. Define a *coSegal 2-algebra* to be a monoid in the category of co-Segal algebras.

Back to Deligne's Conjecture. We wanted to find a resolution $BA \overset{\simeq}{\to} A$ with some map $BA\otimes BA \overset{\simeq}{\to} BA$. Since $BA \overset{\simeq}{\to} A$ is a projective resolution, $BA\otimes BA \overset{q\otimes Id}{\to} BA\otimes A \simeq BA$ is a weak equivalence in $(chBimod_A, \otimes_A, A)$. Indeed, this classical result from homological algebra was perhaps the motivation for Hovey's definition of monoidal model category and his use of the condition that 'cofibrant objects are flat.'

Theorem 4.2. *If K is a field and A is a K-algebra then there are two coSegal algebra structures on $HH^1(A)$.*

We may now state the main result, which relates this work to Deligne's Conjecture:

Theorem 4.3. *Let $\mathcal{M}$ be a monoidal model category satisfying the monoid axiom. Then $REnd(I)$ is a coSegal 2-algebra.*

There is also a version of this theorem for $\mathcal{M}$ an abelian category with enough projectives and injectives.

Kock and Toën approach Deligne's Conjecture by an adjunction between E_∞-algebras and Δ^n-algebras. So one area for future work is to lift their approach on derived mapping spaces to internal hom spaces, i.e. to get a similar result with $REnd(E)$.