WHAT IS AN ELEMENTARY HIGHER TOPOS?

ANDRÉ JOYAL

ABSTRACT. There should be a notion of elementary higher topos in higher topos theory, like there is a notion of elementary topos in topos theory. We are proposing axioms partly inspired by homotopy type theory. We also give a purely categorical description of homotopy type theory.

This talk will contain a definition, but the speaker is hoping to find the best definition possible. We may not get through all the slides, so the reader is encouraged to look at the slides.

Whitehead wanted a purely algebraic theory which is equivalent to homotopy theory. Some approaches so far:

Triangulated categories (Verdier 1963)
Homotopical algebra (Quillen, 1967)
Fibration categories (Brown, 1973)
homotopy theories (Heller, 1988)
theory of derivators (Grothendieck 1987)
Homotopy Type Theory
Elementary higher topos?

The formal language of type theory can be very unnatural to a classically trained mathematician. In particular, the types of proofs which appear. That's why the speaker has attempted to re-frame the theory in category theoretic terms.

See the slides for resources to learn HoTT.

1. CATEGORICAL HOMOTOPY TYPE THEORY

This theory is based on the notion of a *tribe*. There are two types of tribe: π-tribe and h-tribe. This notion should lead to

A quadrable object in a category is one where you can take it's cartesian object with any other object. A map is quadrable if the base change (pullback) of this map along any other map exists.

A *tribe structure* on a category C (with a terminal object $\top$) is a class of maps $\mathscr{F}$ satisfying fibration-like axioms:

Date: April 8, 2014. Prepared by David White, Wesleyan University.

(1) $\mathscr{F}$ contains the isomorphisms and is closed under composition.

(2) Every map in $\mathscr{F}$ is quadrable and $\mathscr{F}$ is closed under base change

(3) The map $X \to \top$ is in $\mathscr{F}$ for all X.

A *tribe* is a category with a terminal object equipped with a tribe structure. A map in $\mathscr{F}$ is called a fibration.

One can view a fibration as a collection of objects parameterized by points in the base, i.e. $p : E \to A$ is a family $(E(x) \ : \ x \in A)$ of objects of C parameterized by elements of A. A tribe is then a collection of families closed under certain operations.

For any object A of a tribe C, the *local tribe* $C(A)$ is the full subcategory of C/A whose objects are the fibrations with codomain A. A map $f : (E, p) \to (F, q)$ is a fibration if $f : E \to F$ is a fibration in C.

Using the language of tribes we now give definitions which are suggestive of a connection to type theory.

An object of a tribe is called a *type*. Notation ⊢ E: Type

A map $t : \top \to E$ is a *term of type E*. Notation ⊢ t:E

An object (E,p) of $C(A)$ is a *dependent type* in *context $x : A$*. Notation x:A ⊢ E(x):Type

A section t of $p : E \to A$ is a *dependent term $t(x) : E(x)$*. Notation x:A ⊢ $t(x)$: $E(x)$

A *homomorphism of tribes* is a functor $F : C \to \mathcal{D}$ which takes fibrations to fibrations, preserves base change along fibrations, and preserves terminal objects. This is like moving from one universe to another via an interpretation.

Example: given any $f : A \to B$ a map in tribe C, the base change functor $f^* : C(B) \to C(A)$ is a homomorphism.

In type theory there are deduction rules, e.g. 'if $y : B \vdash E(y)$:Type' then you can deduce 'x:A ⊢ $E(f(x))$:Type'

This is written

$$\frac{\text{y:B} \vdash \text{E(y):Type}}{\text{x:A} \vdash \text{E(f(x)):Type}}$$

An example of this is context weakening

$$\frac{\vdash \text{E:Type}}{x : A \vdash \text{E:Type}}$$

In the tribe setting this is just the base change functor $i_A : C \to C(A)$ along the map $A \to \top$ which crushes A. We say i_A is *simple* and that $C(A)$ is a *simple extension* of C.

Consider $\delta_A : A \to A \times A$ and you get a map $\delta_A : \top_A \to i_A(A)$ in $C(A)$. It is a term $\delta_A : i_A(A)$, i.e. a term of type $i_A(A)$. This term did not exist in C but does in $C(A)$.

Theorem 1.1. *The simple extension i_A is freely generated by the term $\delta_A : i(A)$, i.e. a term of type A. Thus, $C(A) = C[x_A]$ with $x_A = \delta_A$ just like adding a free variable to a polynomial ring.*

Hence, the diagonal $\delta_A : i(A)$ is generic.

1.1. **Sums and Products.** The base-change functor along a fibration f has a left adjoint, given by composition with f. Denote this $\Sigma_f : C(A) \to C(B)$. Get $\Sigma_A : C(A) \to C$ as the left adjoint of i_A. Then $\Sigma_A(E, p) = E$. So this Σ_A is a summation operator, since the domain of a fibration p is the sum of its fibers. We now have what we need to do dependent types in type theory. In particular, we have the Σ-formation rule

$$\frac{x : A \vdash \text{E(x):Type}}{\vdash \Sigma_{x \in A} E(x) : Type}$$

There is also the Σ-introduction rule, which is how you get a term (a, y) of type $\Sigma_{x:A} E(x)$. You can deduce such a term from the rule above along with $a : A$ and $y : E(a)$.

The section space or product of a map $p : E \to A$ is an object $\prod_A(E) = \prod_A(E, p)$ equipped with $\epsilon : \prod_A(E) \times A \to E$ in C/A called the evaluation, such that for every object C and every map $u : C \times A \to E$ in C/A there is a unique map $v : C \to \prod_A(E)$ such that $v \times A : C \times A \to \prod_A(E) \times A$ is a lift of u along the evaluation map. We write $v = \lambda^A(u)$.

Note: you need the codomain to be quadrable so that ϵ can be defined on a product.

1.2. **π-tribes.** Let f be a quadrable map in C. The product $\Pi_f(E)$ of $E = (E, p) \in C/A$ along $f : A \to B$ is the space of sections of the map $(E, fp) \to (A, f)$ in C/B. For every $y : B$ we have $\Pi_f(E)(y) = \prod_{f(x)=y} E(x)$

A tribe is said to be a *π-tribe* if every fibration $E \to A$ has a product along every fibration $f : A \to B$ and the structure map $\prod_f(E) \to B$ is a fibration. So it's a tribe with products.

In type theory there are $\prod$-formation rule and $\prod$-introduction rules to create the type $\vdash \prod_{x:A} E(x) : Type$ (this is formed by just taking the product over all $x : A$) and terms $\lambda(x : A)t(x)$ of type $\prod_{x:A} E(x) : Type$.

1.3. **h-tribes.** A map $u : A \to B$ is anodyne if it has the left lifting property with respect to every fibration. Think of this as a trivial cofibration.

Say that a tribe C is homotopical (a.k.a. an *h-tribe*) if every map can be factored into an anodyne map followed by a fibration and if the base change of an anodyne map along a fibration is an anodyne map.

We can now define path objects in a way completely analogous to what is done for model categories.

We can form the *identity type* of A as $x : A, y : A \vdash Id_A(x, y) : Type$. A term p of this type is a *proof* that $x = y$. This is one of the big punchlines of type theory. Proofs are terms. Classically, proofs are abstraction. They are not themselves objects.

For example, the proof that $x = x$ is given by the *reflexivity term* $r(x) : Id_A(x, x)$.

Awodey and Warren proved that the reflexivity term $r : A \to Id_A$ is anodyne and so the identity type Id_A is a path object for A, i.e. reflexivity is a lift of the diagonal $A \to A \times A$. Note that this reflexivity term is not unique, but it is unique up to a contractible choice.

The *J-rule* is an operation which takes a commutative square with p a fibration to a diagonal filler $d = J(u, p)$. This was constructed classically in an ad hoc way, but you could perhaps also construct it as a term of type $\mathscr{F} \vdash \mathscr{X}$ where $\mathscr{F}$ is the type of all square shaped diagrams with equality as the bottom horizontal map.

2. Homotopy

The type theorists think of a *homotopy* as a proof that $f = g$, i.e. as a term of type $Id_B(f(x), g(x))$.

This is a small philosophical difference to the work we do, and he hopes this will not prevent homotopy theorists from being friends with type theorists.

With this notion we have the *homotopy category* as a quotient category by the congruence relation.

$u : A \to B$ is homotopy monic if it's part of a homotopy pullback.

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow[u]{} & B \end{array}$$

An object $A \in C$ is an h-proposition if the map $A \to \top$ is homotopy monic. A zero type (or h-set) is an object A where $P(A)$ is an h-proposition, where P is for path object. This occurs iff the diagonal is homotopy monic.

An object is *homotopy initial* if every fibration to this object has a section. They don't ask that this section is unique, but you can show that it's homotopy unique.

There is also a notion of homotopy coproduct as $A \to A \cup B \leftarrow B$ such that any fibration $E \to A \cup B$ and every pair $f, g : A, B \to E$ one can find a section of p to make the diagram commute. Again, this section is only homotopy unique.

A homotopy natural number object is a homotopy initial object in a certain category equipped with an automorphism and a point (think of these as the successor map and the 0 object).

2.1. **Function Extensionality.**

A *Martin-Lof tribe* is an h-tribe which is also a π-tribe such that $\prod_f : C(A) \to C(B)$ preserves the homotopy relation for $f : A \to B$. This is equivalent to the *function extensionality* axiom in the type theory.

A class of small fibrations in a tribe C is a class $\mathscr{F}'$ which contains the isomorphisms and is closed under composition and base change. A small fibration q is universal if every small fibration is a base change of q.

A π-tribe is π-closed if the product of small fibrations and small fibrations is small.

It's called h-closed if the path fibration can be chosen small for all A

A Martin-Lof (ML) universe is C which is π-closed and h-closed.

2.2. **Univalence.**

In a πh-tribe there is an object $Eq(X, Y)$ which represents the homotopy equivalences $X \to Y$.

For every $p : E \to A$ in a πh-tribe there is a fibration $(s, t) : Eq_A(E) \to A \times A$ defined by $Eq_A(E) = Eq(p_1^* E, p_2^* E)$.

A fibration $E \to A$ is *univalent* if the unit map $u : A \to Eq_A(E)$ is a homotopy equivalence.

A Kan fibration is univalent iff it is *uncompressable*. To compress a Kan fibration p is to find a homotopy pullback square expressing p as the base change along a map s which is homotopy surjective but not homotopy monic.

Note: every Kan fibration is the pullback of an uncompressible fibration along a homotopy surjection. Moreover, the compressed fibration is homotopy unique.

Voevodsky proved that the tribe of Kan complexes admits a univalent ML-universe $U' \to U$.

A *Voevodsky tribe* is a ML-tribe C equipped with a univalent ML-universe $U' \to U$. This results in V-type theory.

Voevodsky conjectured that $\vdash s = t : A$ is decidable in V-type theory. Moreover, every globally defined term $\vdash t : \mathbb{N}$ is definitionally equal to a numeral $s''(0) : \mathbb{N}$.

The HoTT group is working on proving this. They're not done yet. If true then you could really work with a computer to do homotopy theory.

See the ending slides for what has been proven in the HoTT.

Open Problems:
No notion of (internal) simplicial object
No notion of (internal) Segal space
No notion of (internal) complete Segal space

An elementary higher topos should be essentially algebraic or combinatorial.

At this point, time ran out. Remaining slide titles:

Grothendieck Topos
Higher topos
Rezk's theorem characterizing higher topos via descent
Toen-Vezzosi characterization
Lurie's characterization
Desiderata for notion of elementary higher topos
Which model of $(\infty, 1)$-category should be used for formalizing it (he used a new notion called pre-model categories, and has a number of slides on them).
Axiomatization for elementary higher topos has 18 axioms coming from a number of places, e.g. homotopical axioms, geometrical axioms, logical axioms, arithmetical axioms

Examples of elementary higher topos: sSet, simplical presheaves over any elegant Reedy category, symmetric cubical sets, one more I didn't catch.

A pre-model category has everything a model category has but not necessarily all colimits and limits. You only have the pullbacks needed to get at homotopy pullback squares with respect to a fibration.

What is an Elementary Higher Topos?

André Joyal

UQÀM

Re-imagining the Foundations of Algebraic Topology,
MSRI April 8, 2014

Plan of the talk

- Review of (categorical) homotopy type theory
- Review of toposes and higher toposes
- Elementary higher topos?

Axiomatic Homotopy Theory

J.H.C. Whitehead (1950):

The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that analytic is equivalent to pure projective geometry.

Examples of axiomatic systems in homotopy theory:

- Triangulated categories (Verdier 1963);
- Homotopical algebra (Quillen 1967);
- Fibration categories (Brown 1973);
- Homotopy theories (Heller 1988)
- Theory of derivators (Grothendieck 198?)
- Homotopy type theory
- Elementary higher topos?

The emergence of Homotopy Type Theory

Gestation:

- **Russell**: *Mathematical logic based on the theory of types* (1908)
- **Church**: *A formulation of the simple theory of types* (1940)
- **Lawvere**: *Equality in hyperdoctrines and comprehension schema as an adjoint functor* (1968)
- **Martin-Löf**: *Intuitionistic theory of types* (1971, 1975, 1984)
- **Hofmann, Streicher**: *The groupoid interpretation of type theory* (1995)

Birth:

- **Awodey, Warren**: *Homotopy theoretic models of identity types* (2006∼2007)
- **Voevodsky**: *Notes on type systems* (2006∼2009)

Suggested readings

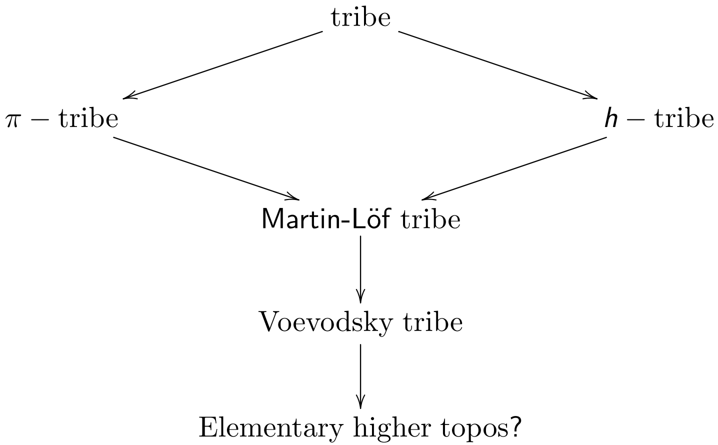
Recent work in homotopy type theory
Slides of a talk by Steve Awodey, AMS meeting January 2014

Notes on homotopy λ-calculus
Vladimir Voevodsky

Homotopy Type Theory
A book by the participants to the Univalent Foundation Program, IAS, 2012-13

Categorical Homotopy Type Theory
Slides of a talk, MIT Topology Seminar, March 17, 2014

Categorical homotopy type theory



Quadrable objects and maps

An object X of a category $\mathcal{C}$ is **quadrable** if the cartesian product $A \times X$ exists for every object $A \in \mathcal{C}$.

A map $p : X \to B$ is **quadrable** if the object (X, p) of the category $\mathcal{C}/B$ is quadrable. This means that the pullback square

$$\begin{array}{ccc} A \times_B X & \xrightarrow{\;p_2\;} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{\;f\;} & B \end{array}$$

exists for every map $f : A \to B$.

The projection p_1 is called the *base change* of $p : X \to B$ *along* $f : A \to B$.

Tribes

Let $\mathcal{C}$ be a category with terminal object $\top$.

Definition

A **tribe structure** on $\mathcal{C}$ is a class of maps $\mathcal{F} \subseteq \mathcal{C}$ satisfying the following conditions:

- $\mathcal{F}$ contains the isomorphisms and is closed under composition;
- every map in $\mathcal{F}$ is quadrable and $\mathcal{F}$ is closed under base changes;
- the map $X \to \top$ belongs to $\mathcal{F}$ for every object $X \in \mathcal{C}$.

A **tribe** is a category $\mathcal{C}$ with terminal object equipped with a tribe structure $\mathcal{F}$. A map in $\mathcal{F}$ is called a **fibration**.

Fibrations and families

The **fiber** $E(a)$ of a fibration $p : E \to A$ at a point $a : A$ is defined by the pullback square

$$
\begin{array}{ccc}
E(a) & \longrightarrow & E \\
\big\downarrow & & \big\downarrow{\scriptstyle p} \\
\top & \xrightarrow{\;a\;} & A.
\end{array}
$$

A fibration $p : E \to A$ is a **family** $(E(x) : x \in A)$ of objects of $\mathcal{C}$ parametrized by a variable element $x \in A$.

A tribe is a collection of families closed under certain operations.

Types and terms

An object E of a tribe $\mathcal{C}$ is called a **type**. Notation:

$$\vdash E : Type$$

A map $t : \top \to E$ in $\mathcal{C}$ is called a **term** of type E. Notation:

$$\vdash t : E$$

The local tribe $\mathcal{C}(A)$

For an object A of a tribe $\mathcal{C}$.

The **local tribe** $\mathcal{C}(A)$ is the full sub-category of $\mathcal{C}/A$ whose objects (E, p) are the fibrations $p : E \to A$ with codomain A.

A map $f : (E, p) \to (F, q)$ in $\mathcal{C}(A)$ is a fibration if the map $f : E \to F$ is a fibration in $\mathcal{C}$.

An object (E, p) of $\mathcal{C}(A)$ is a **dependent type** in **context** $x : A$.

$$x : A \vdash E(x) : Type$$

A section t of a fibration $p : E \to A$ is a **dependent term** $t(x) : E(x)$ in **context** $x : A$.

$$x : A \vdash t(x) : E(x)$$

Homomorphism of tribes

A **homomorphism** of tribes is a functor $F : \mathcal{C} \to \mathcal{D}$ which

- takes fibrations to fibrations;
- preserves base changes of fibrations;
- preserves terminal objects.

For example, if $f : A \to B$ is a map in a tribe $\mathcal{C}$, then the base change functor

$$f^{\star} : \mathcal{C}(B) \to \mathcal{C}(A)$$

is a homomorphism of tribes.

Remark: The category of tribes is a 2-category, where a 1-cell is a homomorphism and 2-cell is a natural transformation.

Base change=change of parameters

In type theory, the base change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ along a map $f : A \to B$ is expressed by the following *deduction rule*

$$\frac{y : B \vdash E(y) : Type}{x : A \vdash E(f(x)) : Type.}$$

In particular, the base change functor $i_A : \mathcal{C} \to \mathcal{C}(A)$ along the map $A \to \top$ takes an object $E \in \mathcal{C}$ to the object $i_A(E) = (E \times A, p_2)$.

The functor i_A is expressed by a deduction rule called *context weakening*:

$$\frac{\vdash E : Type}{x : A \vdash E : Type.}$$

We shall say that the extension $i_A : \mathcal{C} \to \mathcal{C}(A)$ of the tribe $\mathcal{C}$ is *simple*.

Simple extensions are free

Let $i = i_A : \mathcal{C} \to \mathcal{C}(A)$ be the base change functor along the map $A \to \top$.

The diagonal $\delta_A : A \to A \times A$ is a section of the projection $p_2 : A \times A \to A$. It defines a term $\delta_A : i(A)$ in $\mathcal{C}(A)$.

Theorem

The simple extension $i : \mathcal{C} \to \mathcal{C}(A)$ is freely generated by the term $\delta_A : i(A)$. Thus, $\mathcal{C}(A) = \mathcal{C}[x_A]$ with $x_A = \delta_A$.

Hence the diagonal $\delta_A : i(A)$ is **generic**.

Summation

The base change functor $i_A : \mathcal{C} \to \mathcal{C}(A)$ has a left adjoint

$$\Sigma_A : \mathcal{C}(A) \to \mathcal{C}$$

which takes a fibration $p : E \to A$ to its domain $E = \Sigma_A(E, p)$.

Intuitively, the domain of a fibration $p : E \to A$ is the sum of its fibers,

$$E = \sum_{x:A} E(x).$$

Hence the functor $\Sigma_A : \mathcal{C}(A) \to \mathcal{C}$ is a summation operation.

Σ-rules

In type theory, the functor $\Sigma_A : \mathcal{C}(A) \to \mathcal{C}$ is constructed from the Σ-*formation* rule,

$$\frac{x : A \vdash E(x) : \mathit{Type}}{\vdash \sum_{x:A} E(x) : \mathit{Type}}$$

A term $t : \sum_{x:A} E(x)$ is a pair $t = (a, y)$ with $a : A$ and $y : E(a)$.

Hence the Σ-*introduction* rule,

$$\frac{\vdash a : A \qquad \vdash y : E(a)}{\vdash (a, y) : \sum_{x:A} E(x)}$$

The projection $pr_1 : \sum_{x:A} E(x) \to A$ is called the *display map*.

Pushforward

More generally, if $f : A \to B$ is a fibration in a tribe $\mathcal{C}$, then the base change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ has a left adjoint

$$\Sigma_f = f_! : \mathcal{C}(A) \to \mathcal{C}(B)$$

obtained by putting $\Sigma_f(E, p) = (E, fp)$.

We have

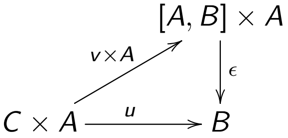
$$\Sigma_f(E)(y) = \sum_{f(x)=y} E(x)$$

for a term $y : B$.

Function space $[A, B]$

Let A be a quadrable object in a category $\mathcal{C}$.

Recall that the **exponential** of an object $B \in \mathcal{C}$ by A is an object $[A, B]$ equipped with a map $\epsilon : [A, B] \times A \to B$, called the *evaluation*, such that for every object $C \in \mathcal{C}$ and every map $u : C \times A \to B$, there exists a unique map $v : C \to [A, B]$ such that $\epsilon(v \times A) = u$.



We write $v = \lambda^A(u)$.

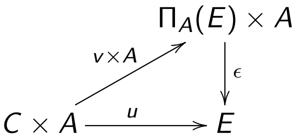
Section space $\Pi_A(E)$

Let A be a quadrable object in a category $\mathcal{C}$.

The **section space** of a map $p : E \to A$ is an object $\Pi_A(E) = \Pi_A(E, p)$ equipped with a map

$$\epsilon : \Pi_A(E) \times A \to E$$

in $\mathcal{C}/A$, called the *evaluation*, such that for every object C and every map $u : C \times A \to E$ in $\mathcal{C}/A$ there exists a unique map $v : C \to \Pi_A(E)$ such that,



We write $v = \lambda^A(u)$.

Product along a map

Let $f : A \to B$ be a quadrable map in a category $\mathcal{C}$.

The **product** $\Pi_f(E)$ of an object $E = (E, p) \in \mathcal{C}/A$ along the map $f : A \to B$ is the space of sections of the map $(E, fp) \to (A, f)$ in the category $\mathcal{C}/B$,

$$\begin{array}{ccc} E & & \Pi_f(E) \\ {\scriptstyle p}\downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

For every $y : B$ we have

$$\Pi_f(E)(y) = \prod_{f(x)=y} E(x)$$

π-tribes

Definition

We say that a tribe $\mathcal{C}$ is π-**closed**, and that it is a π-**tribe**, if every fibration $E \to A$ has a product along every fibration $f : A \to B$ and the structure map $\Pi_f(E) \to B$ is a fibration.

If $f : A \to B$ is a fibration in a π-tribe $\mathcal{C}$, then the base change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ admits a right adjoint

$$\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B).$$

Remark: if $\mathcal{C}$ is a π-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

Examples of π-tribes

- A cartesian closed category, where a fibration is a projection;
- A locally cartesian category if every map is a fibration;
- The category of small groupoids **Grpd**, if a fibration is an iso-fibration (Hofmann and Streicher);
- The category of Kan complexes **Kan**, where a fibrations is a Kan fibration (Streicher, Voevodsky).

Π-rules

In type theory, the functor $\Pi_A : \mathcal{C}/A \to \mathcal{C}$ is constructed from the Π-*formation* rule,

$$\frac{x : A \vdash E(x) : Type}{\vdash \prod_{x:A} E(x) : Type}$$

A term $t : \Pi_{x:A} E(x)$ is a map $x \mapsto t(x)$, where $t(x)$ is a term of type $E(x)$ for each $x : A$.

Hence the Π-*introduction* rule,

$$\frac{x : A \vdash t(x) : E(x)}{\vdash \lambda x t(x) : \prod_{x:A} E(x)}$$

where $\lambda x t(x)$ stands for the map $x \mapsto t(x)$.

Anodyne maps

Definition

We say that a map $u : A \to B$ in a tribe $\mathcal{C}$ is **anodyne** if it has the left lifting property with respect to every fibration $f : X \to Y$.

Thus, if u is anodyne and f is a fibration, then every commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{b} & Y \end{array}$$

has a diagonal filler $d : B \to X$ ($du = a$ and $fd = b$).

Homotopical tribes

Definition

We say that a tribe $\mathcal{C}$ is **homotopical**, or that it is a *h*-**tribe**, if the following two conditions are satisfied

- every map $f : A \to B$ admits a factorization $f = pu$ with u an anodyne map and p a fibration;
- the base change of an anodyne map along a fibration is anodyne.

Remark: if $\mathcal{C}$ is a h-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

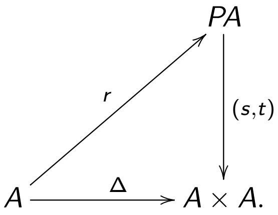
Examples of *h*-tribes

- The category of groupoids **Grpd**, where a functor is anodyne if it is a monic equivalence (Hofmann and Streicher);
- The category of Kan complexes **Kan**, where a map is anodyne if it is a monic homotopy equivalence (Streicher, Awodey and Warren, Voevodsky);
- The syntactic category of Martin-Löf type theory, where a fibration is a map isomorphic to a display map (Gambino and Garner).

Path object

Let A be an object in a h-tribe $\mathcal{C}$.

A **path object** for A is a factorisation of the diagonal $\Delta : A \to A \times A$ as an anodyne map $r : A \to PA$ followed by a fibration $(s, t) : PA \to A \times A$,



Identity type

In Martin-Löf type theory, there is a type constructor which associates to every type A a dependent type

$$x : A, y : A \ \vdash Id_A(x, y) : Type$$

called the **identity type** of A.

A term $p : Id_A(x, y)$ is regarded as a **proof** that $x = y$.

The tautological proof that $x = x$ is given by a term

$$x : A \ \vdash r(x) : Id_A(x, x)$$

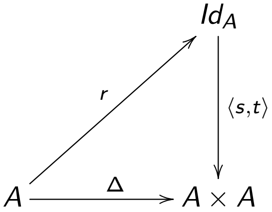
called the **reflexivity term**.

Identity type as a path object

Awodey and Warren: it follows from the J-rule of type theory that the identity type

$$Id_A = \sum_{x:A} \sum_{y:A} Id_A(x, y)$$

is a path object for A,



The J-rule

The J-rule of type theory is an operation which takes a commutative square

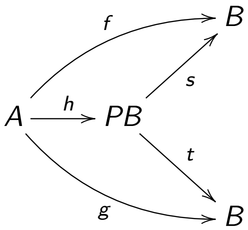
$$\begin{array}{ccc} A & \xrightarrow{u} & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ Id_A & = & Id_A \end{array}$$

with p a fibration, to a diagonal filler $d = J(u)$.

Homotopic maps

Let $\mathcal{C}$ be a h-tribe.

A **homotopy** $h : f \rightsquigarrow g$ between two maps $f, g : A \to B$ in $\mathcal{C}$ is a map $h : A \to PB$



such that $sh = f$ and $th = g$.

In type theory, h is regarded as a **proof** that $f = g$,

$$x : A \vdash h(x) : Id_B(f(x), g(x)).$$

The homotopy category

Let $\mathcal{C}$ be a *h*-tribe.

Theorem

The homotopy relation $f \sim g$ is a congruence on the arrows of $\mathcal{C}$.

The **homotopy category** $Ho(\mathcal{C})$ is the quotient category $\mathcal{C}/\sim$.

A map $f : X \to Y$ in $\mathcal{C}$ is called a **homotopy equivalence** if it is invertible in $Ho(\mathcal{C})$.

Every anodyne map is a homotopy equivalence.

An object X is **contractible** if the map $X \to \top$ is a homotopy equivalence.

h-propositions

A map $u : A \to B$ is *homotopy monic* if the square

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow{u} & B \end{array}$$

is homotopy pullback.

Definition

An object $A \in \mathcal{C}$ is a *h-proposition* if the map $A \to \top$ is homotopy monic.

An object A is a h-proposition iff the diagonal $A \to A \times A$ is a homotopy equivalence.

n-types

The fibration $\langle s, t \rangle : PA \to A \times A$ is an object $P(A)$ of the local tribe $\mathcal{C}(A \times A)$.

An object A is

- a 0-**type** if $P(A)$ is a h-proposition in $\mathcal{C}(A \times A)$;
- a $(n+1)$-**type** if $P(A)$ is a n-type in $\mathcal{C}(A \times A)$.

A 0-type is also called a *h-set*.

An object A is a *h-set* iff the diagonal $A \to A \times A$ is homotopy monic.

Homotopy initial objects

Let $\mathcal{C}$ be a *h*-tribe.

An object $\bot \in \mathcal{C}$ is *h*-**initial** if every fibration $p : E \to \bot$ has a section $\sigma : \bot \to E$,

$$\begin{array}{c} E \\ p \downarrow \; \Big) \uparrow \sigma \\ \bot . \end{array}$$

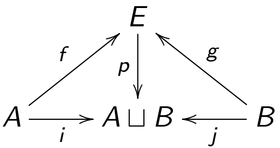
A *h*-initial object remains initial in $Ho(\mathcal{C})$.

Homotopy coproducts

An object $A \sqcup B$ equipped with a pair of maps

$$A \xrightarrow{i} A \sqcup B \xleftarrow{j} B$$

is a h-**coproduct** of A and B if for every fibration $p : E \to A \sqcup B$ and every pair of maps $f, g : A, B \to E$ such that $pf = i$ and $pg = j$,



there exists a section $\sigma : A \sqcup B \to E$ of p such that $\sigma i = f$ and $\sigma j = g$.

A h-coproduct remains a coproduct in $Ho(\mathcal{C})$.

Homotopy natural number object

A *homotopy natural number object* $(\mathbb{N}, s, 0)$ is h-initial in the category of triples (X, f, a), where $X \in \mathcal{C}$, $f : X \to X$ and $a : X$.

If $p : X \to \mathbb{N}$ is a fibration such that $pf = sp$ and $p(a) = 0$

$$\begin{array}{ccccc} \star & \xrightarrow{a} & X & \xrightarrow{f} & X \\ \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \star & \xrightarrow{0} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \end{array}$$

then p has a section $\sigma : \mathbb{N} \to X$ such that $\sigma s = f\sigma$ and $\sigma(0) = a$.

A homotopy natural number object $(\mathbb{N}, s, 0)$ is not necessarily a natural number object in $Ho(\mathcal{C})$.

Homotopy pushout

The notion of homotopy pushout can be defined in a *h*-tribe.

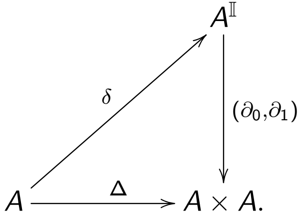
The existence of homotopy pushouts can be added as an axiom (existence of higher inductive types), from which the following objects can be constructed:

- An interval object $(\mathbb{I}, 0, 1)$.
- The join $X \star Y$ of two objects X and Y.
- The *n*-sphere $\mathbb{S}^n$ for every $n \geq 0$.

Interval and path spaces

(Warren) It would be great to extend type theory (or the notion of h-tribe) by adding an interval object $(\mathbb{I}, 0, 1)$ and the following axioms:

- the exponential $A^{\mathbb{I}}$ exists for every type A;
- the joint projection $(\partial_0, \partial_1) : A^{\mathbb{I}} \to A \times A$ is a fibration;
- diagonal $\delta : A \to A^{\mathbb{I}}$ is anodyne.



Martin-Löf tribes

Definition

We say that a h-tribe $\mathcal{C}$ is **Martin-Löf** if it is a π-tribe and the product functor

$$\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$$

preserves the homotopy relation for every fibration $f : A \to B$.

The condition that the functor $\Pi_A : \mathcal{C}(A) \to \mathcal{C}$ preserves the homotopy relation is called *function extensionality*:

$$\frac{x : A \vdash h(x) : f(x) \sim g(x)}{\vdash \lambda x h(x) : f \sim g}$$

Small fibrations and universes

A class of **small fibrations** in a tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is a class of maps $\mathcal{F}' \subseteq \mathcal{F}$ which contains the isomorphisms and is closed under composition and base changes. An object $X \in \mathcal{C}$ is **small** if the fibration $X \to \top$ is small.

A small fibration $q : U' \to U$ is **universal** if for every small fibration $p : E \to A$ there exists a cartesian square:

$$\begin{array}{ccc} E & \longrightarrow & U' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \longrightarrow & U. \end{array}$$

A **universe** is the codomain of a universal small fibration $U' \to U$.

h-universes

The following weaker notion of universality is often sufficient for many purposes.

A small fibration $q : U' \to U$ is h-**universal** if for every small fibration $p : E \to A$ there exists a homotopy cartesian square:

$$\begin{array}{ccc} E & \longrightarrow & U' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \longrightarrow & U. \end{array}$$

Martin-Löf universes

We say that a universe $U' \to U$ in a π-tribe $\mathcal{C}$ is π-**closed** if the product of a small fibration along a small fibration is small.

We say that a universe $U' \to U$ in a h-tribe $\mathcal{C}$ is h-**closed** if the relative path fibration $P_A(E) \to E \times_A E$ can be chosen small for every small fibration $E \to A$.

We shall say that a universe which is both π-closed and h-closed is a **Martin-Löf universe**.

Univalent fibrations

In a πh-tribe $\mathcal{C}$,

For every pair of objects $X, Y \in \mathcal{C}$, there is an object $Eq(X, Y)$ which represents the homotopy equivalences $X \to Y$.

If $p : E \to A$ is a fibration in $\mathcal{C}$, then the fibration

$$(s, t) : Eq_A(E) \to A \times A$$

defined by putting $Eq_A(E) = Eq(p_1^\star E, p_2^\star E)$ represents the homotopy equivalences between the fibers of $p : E \to A$.

Voevodsky:

Definition

A fibration $E \to A$ is **univalent** if the unit map $u : A \to Eq_A(E)$ is a homotopy equivalence.

Uncompressible fibrations

We remark that a Kan fibration is univalent iff it is **uncompressible**.

To *compress* a Kan fibration $p : X \to A$ is to find a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \xrightarrow{s} & B \end{array}$$

in which s is homotopy surjective but not homotopy monic.

Every Kan fibration $X \to A$ is the pullback of an uncompressible fibration $X' \to A'$ along a homotopy surjection $A \to A'$. Moreover, the compressed fibration $X' \to A'$ is homotopy unique.

Voevodsky tribes

Voevodsky: The tribe of Kan complexes **Kan** admits a univalent ML-universe $U' \to U$.

Definition

A **V-tribe** is a *ML*-tribe $\mathcal{C}$ equipped with a univalent *ML*-universe $U' \to U$.

Voevodsky's conjecture : The relation $\vdash s = t : A$ is decidable in V-type theory. Moreover, every globally defined term $\vdash t : \mathbb{N}$ is definitionally equal to a numeral $s^n(0) : \mathbb{N}$.

Results and problems in Hott

Some results:

- Shulman: $\pi_1(\mathbb{S}^1) = \mathbb{Z}$
- Licata: $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ for $n > 0$, $\pi_k(\mathbb{S}^n) = 0$ for $k < n$,
- Brunerie $\pi_3(\mathbb{S}^2) = \mathbb{Z}$
- Lumsdane, Finster, Licata: Freudenthal suspension theorem.

Some problems:

- $\pi_4(\mathbb{S}^3) = \mathbb{Z}/2$?
- No notion of (internal) simplicial object
- No notion of (internal) Segal space
- No notion of (internal) complete Segal space

Beyond type theory?

Type theory is not yet able to support the theory of $(\infty, 1)$-categories.

In which direction should it evolve?

Type theory may be used as an internal language in the theory of higher topos.

But what is an elementary higher topos?

A missing stone

Higher topos	?
Grothendieck topos	Elementary topos

What is an elementary higher topos?

elementary = essentially algebraic = combinatorial

Grothendieck topos

Grothendieck:

Definition

A *topos* is a left exact localization of the category of presheaves $Psh(\mathcal{C})$ on a small category $\mathcal{C}$.

Giraud:

Theorem

A presentable category $\mathcal{E}$ is a topos iff the following conditions are satisfied:

- *colimits are stable by base changes;*
- *coproducts are disjoint;*
- *every equivalence relation is effective.*

Elementary topos

Lawvere and Tierney:

Definition

An *elementary topos* is a locally cartesian closed category with a sub-object classifier (Ω, t).

$t : 1 \to \Omega$ and for every monomorphism $A' \to A$ there exists a unique map $f : A \to \Omega$, such that $f^{-1}(t) = A'$,

$$\begin{array}{ccc} A' & \longrightarrow & 1 \\ \downarrow & & \downarrow t \\ A & \xrightarrow{f} & \Omega \end{array}$$

$\Omega = \{0, 1\}$ in the category **Set**.

We may suppose the existence of a natural number object $\mathbb{N}$.

Geometrical versus logical

If $\mathcal{E}$ and $\mathcal{E}'$ are Grothendieck toposes, then a functor $F : \mathcal{E} \to \mathcal{E}'$ is a *geometric homomorphism* if it preserves

- all (small) colimits
- finite limits

If $\mathcal{E}$ and $\mathcal{E}'$ are elementary toposes, then a functor $F : \mathcal{E} \to \mathcal{E}'$ is a *logical homomorphism* if it preserves

- finite limits;
- internal products: $F\Pi_f(X) \simeq \Pi_{F(f)}(FX)$ for every $f : A \to B$ and every $X \in \mathcal{E}/A$;
- subobject classifiers: $F\Omega \simeq \Omega'$.

Higher topos

Higher topos=homotopy topos=model topos=∞-topos

Rezk:

Definition

A *homotopy topos* is a homotopy left exact Bousfield localization of the model category of simplicial presheaves $sPsh(\mathcal{C})$ on a small simplicial category $\mathcal{C}$.

Lurie:

Definition

An ∞-*topos* is a left exact localization of the quasi-category of pre-stack $Pst(\mathcal{C})$ on a small quasi-category $\mathcal{C}$.

Rezk's theorem

Theorem

A presentable model category $\mathcal{E}$ is a homotopy topos iff it has descent.

descent: for any small diagram $A : I \to \mathcal{E}$ the pullback functor

$$\mathcal{E}/hocolim_I A \to Equifib(\mathcal{E}^I/A)$$

is an equivalence of model categories.

A diagram $B : I \to \mathcal{E}$ over A is equifibered if all the naturality squares

$$\begin{array}{ccc} B(i) & \longrightarrow & B(j) \\ \downarrow & & \downarrow \\ A(i) & \longrightarrow & a(j) \end{array}$$

are homotopy pullback.

Toën and Vezzosi's characterisation

Theorem

A presentable model category $\mathcal{E}$ is a higher topos iff the following conditions are satisfied:

- *colimits are stable under base changes;*
- *coproducts are disjoint;*
- *every Segal groupoid is the Cech complex of a map.*

The *Cech complex* of a map $f : A \to B$ is the simplicial object $C_\star(f)$ defined by putting

$$C_n(f) = A \times_B \ldots \times_B A \quad (n + 1 \text{ times})$$

for every $n \geq 0$.

Lurie's characterisation

Theorem

A presentable quasi-category $\mathcal{E}$ is a higher topos iff its class of k-compact morphisms has a classifying universe $U'_k \to U_k$ for each regular cardinal $k > 0$.

Thus, for every k-compact morphism $A' \to A$, there exists a pullback square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & U'_k \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & U_k \end{array}$$

and the pair (f, f') is *homotopy unique*.

Desiderata for the notion of EH-topos

A wish list:

- The notion of EH-topos should be essentially algebraic;
- Every slice $\mathcal{E}/A$ of an EH-topos $\mathcal{E}$ should be an EH-topos;
- The category of *internal* simplical objects $\mathcal{E}^{\Delta^{op}}$ of an EH-topos $\mathcal{E}$ should be an EH-topos;
- Every higher topos should be *equivalent* to an EH-topos.

Which is best?

The notion of EH topos could be formalized by using any notion of $(\infty, 1)$-category:

- simplicial category
- complete Segal space;
- Segal category;
- model category;
- relative category.
- quasi-category

These notions are equivalent (Bergner, Joyal-Tierney, Barwik-Kan), but very different algebraically.

The language of Quillen model categories seems best for formulating the notion of EH-topos.

Wild objects in a model category

But the classical notion of model category is *too permissive* for axiomatic homotopy theory.

Because an object $A \times_B C$ defined by a pullback square

$$\begin{array}{ccc} A \times_B C & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is not homotopy invariant, unless the square is homotopy pullback.

Ungrammatical sentences are normally excluded from a formal language.

We need to generalize the notion of model category.

pre-model category

Lumsdane, Stanculescu:

Definition

A *pre-model structure* on a category $\mathcal{E}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes maps in $\mathcal{E}$ such that:

- $\mathcal{W}$ satisfies 3-for-2;
- Each pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a weak factorization systems;
- maps in $\mathcal{F}$ are quadrable and maps in $\mathcal{C}$ co-quadrable.

A *pre-model category* is a category $\mathcal{E}$ with $\top$ and $\bot$ equipped with a pre-model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

Properness

A pre-model structure is *right proper* (resp. *left proper*) if the base (resp. cobase) change of a weak equivalence along a fibration (resp. a cofibation) is a weak equivalence. A pre-model structure is *proper* if it is both left and right proper.

EH-topos?

EH-topos?

Tentative definition (version 2).

Definition

We shall say that a pre-model category $\mathcal{E}$ equipped a universe $U' \to U$ is an **elementary higher topos** if it satisfies the following axioms:

EH-topos?

Tentative definition (version 2).

Definition

We shall say that a pre-model category $\mathcal{E}$ equipped a universe $U' \to U$ is an **elementary higher topos** if it satisfies the following axioms:

- Homotopical axioms: H1-H3

EH-topos?

Tentative definition (version 2).

Definition

We shall say that a pre-model category $\mathcal{E}$ equipped a universe $U' \to U$ is an **elementary higher topos** if it satisfies the following axioms:

- Homotopical axioms: H1-H3
- Geometrical axioms G1-G7

EH-topos?

Tentative definition (version 2).

Definition

We shall say that a pre-model category $\mathcal{E}$ equipped a universe $U' \to U$ is an **elementary higher topos** if it satisfies the following axioms:

- Homotopical axioms: H1-H3
- Geometrical axioms G1-G7
- Logical axioms L1-L6

EH-topos?

Tentative definition (version 2).

Definition

We shall say that a pre-model category $\mathcal{E}$ equipped a universe $U' \to U$ is an **elementary higher topos** if it satisfies the following axioms:

- Homotopical axioms: H1-H3
- Geometrical axioms G1-G7
- Logical axioms L1-L6
- Arithmetical axioms A1-A2

EH-topos?

Tentative definition (version 2).

Definition

We shall say that a pre-model category $\mathcal{E}$ equipped a universe $U' \to U$ is an **elementary higher topos** if it satisfies the following axioms:

- Homotopical axioms: H1-H3
- Geometrical axioms G1-G7
- Logical axioms L1-L6
- Arithmetical axioms A1-A2

There are 18 axioms in all!

EH-topos?

Tentative definition (version 2).

Definition

We shall say that a pre-model category $\mathcal{E}$ equipped a universe $U' \to U$ is an **elementary higher topos** if it satisfies the following axioms:

- Homotopical axioms: H1-H3
- Geometrical axioms G1-G7
- Logical axioms L1-L6
- Arithmetical axioms A1-A2

There are 18 axioms in all!

Remark: Hilbert's elementary euclidian geometry has 19 axioms.

Homotopical axioms

- H1: the model structure is right proper
- H2: every object is cofibrant
- H3: the base change of a cofibration along a fibration is a cofibration

Remark: It follows from these axioms that the pre-model structure is proper.

We denote by $\mathcal{E}(A)$ the full subcategory of $\mathcal{E}/A$ whose objects are the fibrations $p : E \to A$ with codomain A.

It follows from the axioms that $\mathcal{E}(A)$ is a h-tribe.

Geometrical axioms

- G1: the initial object $\bot$ is strict and the map $\bot \to X$ is a fibration for every X;
- G2: the inclusions $X \to X \sqcup Y$ is a fibration for every pair of objects (X, Y);
- G3: the functor $(i_1^\star, i_2^\star) : \mathcal{E}/(X \sqcup Y) \to \mathcal{E}/X \times \mathcal{E}/Y$ is an equivalence of pre-model categories.
- G4: if $f : X \to Y$ is a fibration, then the base change functor $f^\star : \mathcal{E}/Y \to \mathcal{E}/X$ preserves cobase changes of cofibrations;
- G5: the contravariant pseudo-functor $\mathcal{E}(-) : \mathcal{E}^{op} \to Cat$ takes the square of a cobase change of a cofibration to a pseudo-pullback;
- G6: every fibration factors as a homotopy surjection followed by a monic fibration;
- G7: every pseudo-groupoid is effective.

Logical axioms

- L1: the product of a fibration along a fibration exists;
- L2: if $u : A \to B$ is a cofibration between fibrant objects and $p : E \to B$ is a fibration, then the map

  $$u^\star : \Pi_B(E) \to \Pi_A(u^\star(E))$$

  induced by u is fibration. Moreover, $u^\star$ is acyclic when u is acyclic.
- L3: the axiom L2 is true in every slice $\mathcal{E}/A$;
- L4: small fibrations are closed under composition;
- L5: the product of a small fibration along a small fibration is small;
- L6: the universe U is fibrant and the fibration $U' \to U$ is univalent.

Arithmetical axioms

- A1: $\mathcal{E}$ contains a natural number object $\mathbb{N}$
- A2: $\mathbb{N}$ is fibrant and small.

Examples of *EH*-topos

- The category of simplicial sets **sSet** (Voevodsky);
- The category of simplicial presheaves over any elegant Reedy category (Shulman).
- The category of symmetric cubical sets (Coquand).
- The category of presheaves over any elegant (local) test category (Cisinski).

Easy consequences

If $\mathcal{E}$ is an *EH*-topos, then

- the category $\mathcal{E}/A$ is an *EH*-topos for every object $A \in \mathcal{E}$;
- the base change functor $f^\star : \mathcal{E}/B \to \mathcal{E}/A$ is a logical homomorphism of *EH*-topos for every fibration $f : A \to B$.
- the functor $f^\star : \mathcal{E}/B \to \mathcal{E}/A$ induces an equivalence of homotopy categories iff the fibration f is acyclic.

Moreover

- the category $\mathcal{E}(A)$ has the structure of a V-tribe for every A;
- the base change functor $f^\star : \mathcal{E}(B) \to \mathcal{E}(A)$ is a homomorphism of V-tribes for every map $f : A \to B$;
- the functor $f^\star : \mathcal{E}(B) \to \mathcal{E}(A)$ induces an equivalence of homotopy categories iff f is acyclic.

Some critiques

- It maybe enough to suppose that the small fibration $U' \to U$ is h-universal;
- the univalence of the fibration $U' \to U$ may not be necessary in the presence of axiom G7.

THANK YOU FOR YOUR ATTENTION!