David Gepner: Thom spectra and twisted umkehr maps

Today we will report on joint work of Ando–Blumberg–G–Hopkins–Rezk on *generalized Thom spectra*, as well as on further work of Ando–Blumberg–G on multiplicative properties and on twisted umkehr maps.

Classically, if we have a group G with a map $G \to GL_1(S) = hAut(S)$, we can form $S_{hG} = S//G$, and this is the Thom spectrum MG. For instance, with $G = \{e\}$ we get $S = MFramed$. More computably, the map $O \to GL_1(S)$ via the action of $O(n)$ on (S^n, ∞) (as a based space) is in fact an E_∞ map, and its associated Thom spectrum is MO. Moreover, O has a family of interesting connective covers, which give rise to interesting Thom spectra

$$M\left(\cdots 5Brane \to String \to Spin \to SO \to O\right).$$

[Cute, David.]

We also have $U \to GL_1(S)$ via the action of $U(n)$ on S^{2n}, and this gives rise to MU. Now, $\pi_* MU \cong L$, the Lazard ring, and this gives rise to connections between formal group laws and complex orientations, which has played a fundamental role in modern algebraic topology.

Now, no foundations have been reimagined yet. But another feature of Thom spectra is that they come up in twisted co/homology, and so we'd like a slightly more robust notion of them that works in other contexts. To do this, we use the formalism of ∞-categories – not only because we like it, but because it adds a lot of flexibility. For instance, certain desired adjoints are difficult to compute in models, but come formally at the level of ∞-categories.

We let $\mathcal{S}$ denote the ∞-category of spaces, i.e. ∞-groupoids following the homotopy hypothesis. (One can obtain a version of $\mathcal{S}$ by taking the nerve of any reasonable category of spaces and then inverting the weak equivalences in the ∞-categorical sense. The main point is that this comes directly from the model category of topological spaces and weak equivalences.) Now, given $X \in \mathcal{S}$, we have the slice ∞-category $\mathcal{S}_{/X}$, and this construction is contravariantly functorial in X via restriction, i.e. pullback. That is, a map $Y \xrightarrow{f} X$ gives $f^* : \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$. One of the most useful things about this is that f^* has both a left adjoint $f_!$ and a right adjoint f_*. Thus, we get a functor

$$\mathcal{S}_{/-} : \mathcal{S}^{op} \to \mathtt{Cat}_\infty.$$

However, because things are nice (read: presentable), this actually factors through the inclusion $\mathtt{Pr}^{L,R} \hookrightarrow \mathtt{Cat}_\infty$ of *presentable* ∞-categories, with morphisms given by those functors that admit both left and right adjoints.

Now, here are some perks of this factorization.

1. The functor $\mathcal{S}_{/-} : \mathcal{S}^{op} \to \mathtt{Cat}_\infty$ satisfies *descent*, i.e. it's a sheaf of ∞-categories on $\mathcal{S}$ with respect to the canonical topology. (One might call this an "$(\infty, 2)$-topos".) More precisely, given $X = \operatorname{colim} X_i$, then $\mathcal{S}_{/X} \xrightarrow{\sim} \lim \mathcal{S}_{/X_i}$. (This is not totally formal.)
2. Each slice $\mathcal{S}_{/X}$ is symmetric monoidal via fibered product (over X); we'll denote this by $\otimes_X$.

Thus, we actually get a sheaf

$$\mathcal{S}_{/-} : \mathcal{S}^{op} \to \mathtt{CAlg}(\mathtt{Pr}^{L,R}).$$

This is a lot of structure; in fact, this all amounts to a "Wirthmüller context" (in the sense of Fausk–Hu–May). Here's another property.

3. Since $\mathcal{S}$ is freely generated under colimits by $* \in \mathcal{S}$, any sheaf $\mathcal{S}^{op} \to \mathtt{CAlg}(\mathtt{Pr}^{L,R})$ is determined uniquely by its value on a single point.

In other words, all of this data is equivalent to a single symmetric monoidal ∞-category $\mathcal{C} \in \mathtt{CAlg}(\mathtt{Pr}^L)$. Namely, given $\mathcal{C}$, for any $X \in \mathcal{S}$ we set $\mathcal{C}_{/X} = \mathtt{Fun}(X^{op}, \mathcal{C})$ (where the "op" is optional since X is an ∞-groupoid – very funny, David) and so we consider this as $\mathtt{Pre}_{\mathcal{C}}(X)$, the category of $\mathcal{C}$-valued presheaves on X. Then, $f^* : \mathtt{Fun}(X^{op}, \mathcal{C}) \to \mathtt{Fun}(Y^{op}, \mathcal{C})$ is visibly symmetric monoidal and admits left and right adjoints.

This leads to a question: How is this related to $\mathcal{S}_{/-}$ in the case that $\mathcal{C} = \mathcal{S}$? Well, for any $X \in \mathcal{S}$, we have that $X \simeq \operatorname{colim}_{* \to X} *$, and so

$$\mathcal{S}_{/X} \simeq \lim_{* \to X} \mathcal{S}_{/*} = \lim_{* \to X} \mathcal{S} = \mathtt{Fun}(X^{op}, \mathcal{S}).$$

This is of course totally dependent on being able to think of X both as a space and as an ∞-groupoid, hence as an ∞-category.

Now, let's get back to the Wirthmüller context. This gives lots of handy formulae. For instance, consider the sheaf

$$\mathcal{C}_{/-} : \mathcal{S}^{op} \to \texttt{CAlg}(\texttt{Pr}^{L,R})$$

determined by the symmetric monoidal presentable ∞-category $\mathcal{C}$ (which is automatically closed, by the adjoint functor theorem). Given $Y \xrightarrow{f} X$, for formal reasons we have things like

$$f^*(M \otimes_X N) \simeq f^*M \otimes_Y f^*N$$

and

$$f_!(f^*M \otimes_Y N) \simeq M \otimes_X f_!N.$$

Moreover, if $\mathcal{C}$ is stable and $Y \xrightarrow{f} X$ is proper (in the weak sense that over each $x \in X$, the fiber Y_x is a compact object), then the right adjoint $f_!$ admits a *further* right adjoint $f^!$. This gives rise to dualizing complexes (in the sense that Vesna Stojanoska told us about earlier this week).

Now, what does this have to do with Thom spectra? Let's specialize to $\mathcal{C} = \texttt{Mod}(A)$, where A is an E_∞-ring spectrum (although this should work for A being only E_n too).

Example 22. With $A = \mathcal{S}$, we get $\mathcal{C} = \texttt{Sp}$, the ∞-category of spectra. Then, by definition we obtain $\texttt{Sp}_{/X} = \texttt{Fun}(X^{op}, \texttt{Sp}) = \texttt{Pre}_{\texttt{Sp}}(X)$, the category of presheaves of spectra on X. This has a unit $\mathbb{S}_X$, given by $p^*\mathbb{S}$ where $X \xrightarrow{p} *$. This category has a fiberwise smash product. If say $X = BG$ (for G some A_∞-group) we get

$$\texttt{Sp}_{/BG} \simeq \texttt{Fun}(BG^{op}, \texttt{Sp}) \simeq \texttt{Mod}(\mathbb{S}[G]).$$

We'll actually stick to this case for concreteness, though it's totally unnecessary. But to continue, note that we have a functor

$$Pic : \texttt{CAlg}(\texttt{Pr}^L) \to \texttt{CAlg}^{gp}(\mathcal{S}) \simeq \texttt{Sp}_{\geq 0},$$

which takes a symmetric monoidal presentable ∞-category to its *Picard ∞-groupoid*, which is the grouplike E_∞-space of tensor-invertible objects in $(\mathcal{C}, \otimes)$. For short, we write $Pic(\mathcal{C}) = \mathcal{C}^\times$.

Theorem 10 (ABG)**.** *The functor Pic has a left adjoint,* `Pre`*, equipped with the Day convolution symmetric-monoidal structure (which uses the multiplication on the space). Moreover, the counit of the adjunction $\mathcal{S}_{/Pic(\mathcal{C})} \to \mathcal{C}$ is the "generalized Thom spectrum" functor, which is colimit-preserving and symmetric-monoidal.*

Let's describe the convolution. We already saw that $\texttt{Pre} \simeq \mathcal{S}_{/-}$, and using this identification we convolve as

$$(Y \to X) \otimes (Z \to X) = (Y \times Z \to X \times X \xrightarrow{\mu} X).$$

Now, we remark again the $\mathcal{C}$ can be taken to be E_n for $n > 0$. Also, for $\mathcal{C} = \texttt{Sp}$, $Pic(\mathcal{C}) = Pic(\mathbb{S}) = \mathbb{Z} \times BGL_1(\mathbb{S})$. So, a map $BG \xrightarrow{\alpha} BGL_1(\mathbb{S})$ deloops to an A_∞ map $G \to GL_1(\mathbb{S})$, and the "generalized Thom spectrum" functor sends α to the Thom spectrum $MG \simeq \mathbb{S}/G$. (More generally, any space X is a coproduct of BG's for G an ∞-group (i.e. a grouplike A_∞-space), and this gets sent to a wedge of Thom spectra.)

Really, we want to restrict to $\mathcal{C} = \texttt{Mod}(A)$ for A an E_∞-ring spectrum. Then, we can define the *A-twisted co/homology* of $X \xrightarrow{\alpha} Pic(A)$ as follows. Write X^α for the corresponding Thom A-module spectrum. Then, we define

$$A_n^\alpha(X) = \pi_0 \mathrm{map}_A(\Sigma^n A, X^\alpha) = \pi_n X^\alpha$$

and

$$A_\alpha^n(X) = \pi_0 \mathrm{map}_A(X^\alpha, \Sigma^n A).$$

Let us describe some interesting examples of this construction.

Example 23. Take $A = KU$. Then, we have $BGL_1(KU) \to Pic(KU)$. Moreover, $GL_1(KU)$ contains a copy of $BU^\otimes$, and this deloops further to give us

$$K(\mathbb{Z}, 3) = BBU(1) \to BBU^\otimes \to BGL_1(KU) \to Pic(KU),$$

the twisted K-theory $K_\alpha^*(X)$ for $X \in \mathcal{S}$ and $\alpha \in H^3(X; \mathbb{Z})$. This works equally well for ku (and there are analogs for KO and ko, but with $K(\mathbb{Z}/2, 2)$ instead).

We can ask if there are any canonical twists for tmf or for $\mathbb{K}(ku)$. In fact, there is a map $K(\mathbb{Z},4)$ into Pic of each of these, which are interesting so we'll describe them now.

Example 24. First of all, we have the connective cover $BString \to BSpin$ by taking the fiber of the map $BSpin \to K(\mathbb{Z},4)$ classifying the lowest-dimensional homotopy of $BSpin$. This composes to a long fiber sequence

$$K(\mathbb{Z},3) \to BString \to BSpin \to K(\mathbb{Z},4).$$

Then, we get a map of Thom spectra

$$\mathbb{S}[K(\mathbb{Z},3)] = \Sigma^\infty_+ K(\mathbb{Z},3) = Th(K(\mathbb{Z},3) \xrightarrow{*} BGL_1(\mathbb{S})) \to Th(BString \to BSpin \to BO \to BGL_1(\mathbb{S}) \simeq MString.$$

Then, the String orientation of Ando–Hopkins–Rezk is a map $MString \to tmf$. By the adjunction, this long composite is equivalent to a map $K(\mathbb{Z},3) \to GL_1(tmf)$, and this deloops and composes to a map

$$K(\mathbb{Z},4) \to BGL_1(tmf) \to Pic(tmf).$$

This gives us twisted elliptic cohomology corresponding to degree-4 integral cohomology elements.

Example 25. Let's proceed to $\mathbb{K}(ku)$. The underlying loopspace is $\mathbb{Z} \times BGL(ku)$, and there's an E_∞-map $K(\mathbb{Z},3) \simeq BK(\mathbb{Z},2) \to BGL_1(ku)$ (via the K-theory orientation that we discussed earlier), and this composes as

$$K(\mathbb{Z},3) \simeq BK(\mathbb{Z},2) \to BGL_1(ku) \to BGL(ku) \to GL_1(\mathbb{K}(ku)) \to Pic(\mathbb{K}(ku)).$$

Example 26. We can explain Atiyah duality in this framework too. If M is a closed compact manifold with tangent bundle T, we have the stable normal bundle $M \xrightarrow{-T} BO$, and this composes to

$$M \xrightarrow{-T} BO \to BGL_1(\mathbb{S}) \to Pic(\mathbb{S}),$$

and then we recover the well-known fact that $M^{-T} \simeq D\Sigma^\infty_+ M$, totally formally from the Wirthmüller context and the properness of $M \to *$.