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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: James Dilts Email/Phone: JDilts@uoregon.edu

Speaker's Name: David Maxwell

Talk Title: The conformal method of constructing Cauchy data for the Einst. eqns.

Date: 9 / 9 / 13 Time: 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Conformal method, Cauchy data, Constraint equations

Please summarize the lecture in 5 or fewer sentances: He reviewed the derivation of the constraint equations for Cauchy data, Then gave a review of the history of the progress in finding solutions, particularly via the conformal method. He talked of the work of Lichnerowicz, Choquet-Bruhat and York up to the work of Isenberg in '95.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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Maxwell - Conformal Method

Fig 1

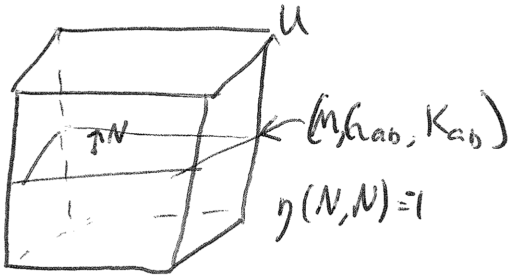
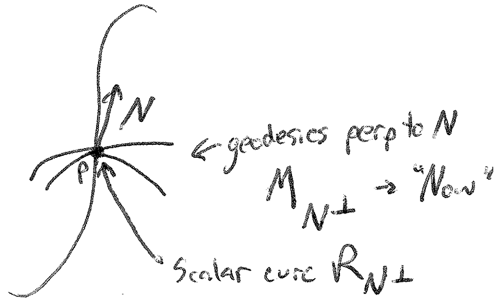


Fig 2



here, K=0 at p.

Fig 3

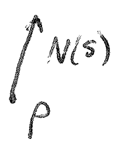


Fig 4

Fig 5

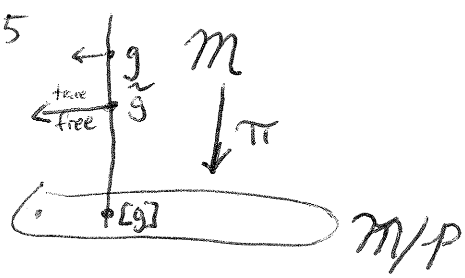
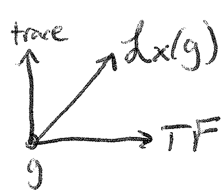


Fig 6



$$\mathcal{L}_X g = \left[\nabla_a X_b + \nabla_b X_a - \tfrac{2}{3}\,\mathrm{div}\,X\, g_{ab}\right] + \tfrac{2}{3}\,\mathrm{div}\,X\, g_{ab}$$

Yamabe class

Fig 7

	−	0	+
$\sigma\not\equiv 0, \tau\not\equiv 0$	✓	✓	✓
$\sigma\equiv 0, \tau\not\equiv 0$	✓	X	X
$\sigma\not\equiv 0, \tau\equiv 0$	X	X	✓
$\sigma\equiv 0, \tau\equiv 0$	X	✓	X

] sol'n not unique, but has homothetic family

THE CONFORMAL METHOD OF CONSTRUCTING CAUCHY DATA FOR THE EINSTEIN EQUATIONS

DAVID MAXWELL

Let (U, η_{ab}) be the spacetime. See fig 1. M is our spacelike slice. We have the field equations $G_{ab} + \Lambda\eta_{ab} = 8\pi T_{ab}$ which we will call (*) and so we have $G(N,N) - \Lambda = 8\pi T(N,N)$, which we will call (**). (*) is equivalent to (**) holding for all timelike unit vectors. $T(N,N)$ is the energy density as observed by N, i.e. an observing going through the point with tangent vector N. $\mathrm{Ric}(N,N)$ is the average of sectional curvatures containing N. Similarly, $G(N,N)$ is the average of sectional curvatures *perpendicular* to N.

The Gauss equation allows us to compute this average. This leads to "Hamiltonian constraint equation", $\frac{1}{2}\left[R_h - |K|_h^2 + (\mathrm{tr}_h K)^2\right] = 8\pi\rho + \Lambda$. (Here, ρ is $T(N,N)$, i.e. the energy density.) Now see Fig 2. If we take geodesics perpendicular to N, and make them together a surface, then this constraint says $\frac{1}{2}R_{N^\perp} = 8\pi\rho + \Lambda$, i.e. the scalar curvature of "now" is equal to the energy density. Or "matter gobbles space" (because there's positive scalar curvature.)

Momentum constraint: Let $N(s)$ be a family of unit timelike vectors through p (see fig 3). We then have $N'(0) = X$ is spacelike and $\eta(N,X) = 0$. So we then have

$$G(N,N) = 8\pi T(N,N) + \Lambda$$
$$G(N,X) = 8\pi T(N,X)$$

We have $T(N,X) = -j_a X^a$, the momentum density. The Codazzi equation then lets us get

$$\nabla^b\left[K_{ab} - (\mathrm{tr}_h K)h_{ab}\right] = 8\pi j_a.$$

We'll now start calling $\mathrm{tr}_h K = \tau$ for simplicity. So now we have $\mathrm{div}K - d\tau = 8\pi j$. This is 4 equations for the constraints (1 for Hamiltonian constraint, 3 for momentum constraint), but 12 unknowns, so the system of constraints is underdetermined. So then, we have the central questions: 1. How do you construct solutions with a given property? 2. How do you parameterize the set of solutions?

The easiest case is when M is compact, vacuum (i.e. $\rho = j = 0$) and $\Lambda = 0$. This case still has all the significant open problems, so we'll focus on this case.

Conformal method: To motivate, let's look for solutions where $\tau \equiv 0$. Such solutions are called maximal, because we could make the slice smaller by wiggling the slice a bit (see Fig 4). Here, we have $R_h - |K|^2 = 0$ and $-\nabla^a K_{ab} = 0$. This tells us $\mathrm{tr}K = 0$ (i.e. traceless), and $\mathrm{div}K = 0$ (called transverse), and so K is transverse-traceless, i.e. is a TT tensor.

Lichnerowicz in 1944: The set of TT tensors behaves nicely under conformal changes, i.e. if we let g_{ab} be a metric, and σ_{ab} be TT with respect to g_{ab} and let $\tilde{g}_{ab} = \phi^4 g_{ab}$ and $\tilde{\sigma}_{ab}\phi^{-2}\sigma_{ab}$, then $\tilde{\sigma}$ is TT w.r.t $\tilde{g}$. Here, the 4 depends on the dimension n (he didn't say, but it is $2^* - 2 = \frac{2n}{n-2} - 2$), but the -2 is always a -2.

He describes an ad hoc method for constructing TT tensors. So, we can construct σ_{ab} given g_{ab}. He then seeks solutions to the constraints of the form $h_{ab} = \phi^4 g_{ab}$ and $K_{ab} = \phi^{-2}\sigma_{ab}$. And so this gives us solutions to the constraints with $\tau \equiv 0$.

The Hamiltonian constraint is satisfied by this data so long as $-8\Delta_g\phi + R_g\phi = |\sigma|_g^2\phi^{-7}$, which is called the Lichnerowicz equation. He worked on a bounded domain, with Dirichlet boundary conditions. He proves 1. uniqueness, 2. if $R_g > 0$, $|\sigma|$ small, then existence, 3. if R_g is anything, but the domain is small, then existence. In his paper he also describes a solution for the n-body problem using irrotational ideal fluid for the bodies.

As we generalize away from $\tau \equiv 0$, TT tensors still play a role.

Notation: $\mathcal{M}$ is the space of Riemannian metrics on M. See Fig 5. This is a Frechet manifold. So for $g \in \mathcal{M}$, $T_g\mathcal{M} = C^\infty S_2(M)$, i.e. symmetric (0,2) tensors over M. $\mathcal{P}$ is the space of smooth positive functions. and $\mathcal{M}/\mathcal{P}$ is then the space of conformal classes. $\mathcal{M}$ admits a canonical Riemannian metric itself; for $l_{ab}, m_{ab} \in T_g\mathcal{M}$ we define $[l_{ab}, m_{ab}] = \int_M \langle l, m\rangle_g dV_g$.

$T_g\mathcal{M}$ admits a trivial decomposition, $T_g\mathcal{M} = Trace(g) \oplus TF(g)$, where $Trace(g)$ is $\{f g_{ab} : f \in C^\infty(M)\}$, i.e. trace parts, and where TF is the trace free matrices. And then $\ker \pi_{*,g} = Trace(g)$, where π is the projection onto conformal classes, and so we can identify $TF(M)$ with $T_{[g]}\mathcal{M}/\mathcal{P}$. Moreover, if $\tilde{l}_{ab} = \phi^4 l_{ab}$, then $\pi_{*,\tilde{g}}(\tilde{l}_{ab}) = \pi_{*,g}(l_{ab})$.

Moral: Symmetric $(0,2)$ that conformally transform according to $\tilde{l}_{ab} = \phi^4 l_{ab}$ (i.e. trace free ones specifically) represent elements of $T_{[g]}\mathcal{M}/\mathcal{P}$, i.e. are tangent matrices to conformal classes. They can represent small changes of a conformal class.

$Flow(g) = \{\mathcal{L}_X g : X \in C^\infty(TM)\}$, i.e. all possible you can get by pulling back by diffeomorphisms, or the set of tangent matrices you can get by flowing by diffeomorphisms in any given direction. See fig 6. $LX_{ab} = \nabla_a X_b + \nabla_b X_a - \frac{2}{3}\text{div}X g_{ab}$ is the conformal killing operator, and is the trace free part of the Lie derivative of g, $\mathcal{L}_x g$, by construction. $CK(g) = \{LX : X \in C^\infty(TM)\}$ is thus the tangent vectors of $\mathcal{M}/\mathcal{P}$. since $L_g X = \phi^4 L_{\tilde{g}} X$. This is the set of metrics we can go through by essentially just changing coordinates.

Suppose $\tilde{\sigma}_{ab} = \phi^{-2}\sigma_{ab}$. If we raise indices with respect to the appropriate metrics, we get $\tilde{\sigma}^{ab} = \phi^{-10}\sigma^{ab}$, $d\tilde{V} = \phi^6 dV$ and $\tilde{\sigma}^{ab} d\tilde{V} = \phi^{-4}\sigma^{ab} dV$. Thus we have $\int \sigma^{ab} l_{ab} dV = \int \tilde{\sigma}^{ab}\tilde{l}_{ab} d\tilde{V}$ where $\tilde{l}_{ab} = \phi^4 l_{ab}$. This tells us that TT tensors (like σ) encode cotangent vectors to the set of conformal classes and so are in some sense conformal momenta of the (conformal) metric.

But not just any cotangent vectors: Consider some $LX \in T_{[g]}\mathcal{M}/\mathcal{P}$. Then notice that we have

$$\langle \sigma^{ab} dV, LX\rangle = \int_M \sigma^{ab} LX_{ab} dV = -2\int \nabla_a \sigma^{ab} x_b dV = 0,$$

by integration by parts. Thus these cotangent vectors annihilate things of the form LX. Thus TT tensors are (conformal) momenta that don't care about diffeomorphisms.

Thus for the conformal method so far, we specify 1. the conformal class of metric and 2. a conformal momentum (that ignores diffeomorphisms).

There were extensions of this technique by Choquet-Bruhat and York in the 70's. In 1971, York shows $\tau \equiv 0$ isn't necessary. In fact, $\tau \equiv c$, a constant, will do. Again, if we specify $(h_{ab}, \sigma_{ab}, \tau)$ for τ constant. We then seek solutions of the form $h_{ab} = \phi^4 g_{ab}$, and $K_{ab} = \phi^{-2}\sigma_{ab} + \frac{\tau}{3}\phi^4 g_{ab}$, so τ really is the mean curvature with respect to the scaled metric h_{ab}. Also, again, the momentum constraint is satisfied automatically, so we just need to worry about the Hamiltonian constraint.

The Lichnerowicz equation (i.e. the Hamiltonian constraint) becomes

$$-8\Delta\phi + R_g\phi = |\sigma|^2\phi^{-7} - \frac{2}{3}\tau^2\phi^5$$

called (1). This is very reminiscent of the Yamabe problem equation,

$$-8\Delta\phi + R\phi = c\phi^5$$

called (2). [The Yamabe problem says that if you can solve (2), then $\phi^4 g$ has scalar curvature c, where c is constant for the Yamabe problem in specific.] 5 is a critical exponent [for the Sobolev embedding; see Lee and Parker's paper "The Yamabe Problem" for an excellent reference.] So the Lichnerowicz equation is like the negative case of the Yamabe problem, which is the easy case. The solvability of (1) and (2) depend on the Yamabe invariant (of the metric g)

$$Y_g = \inf_{\phi>0} \frac{\int_M 8|\nabla\phi|^2 + R\phi^2 dV}{\|\phi\|^2_{L^6}}.$$

The Yamabe invariant is negative if and only if there is a $\tilde{g} \in [g]$ such that $R_{\tilde{g}} < 0$, or equivalently, $R_{\tilde{g}} = -1$. Similar statements hold for $Y_g = 0$ and $Y_g > 0$, though they are harder to prove.

We can rewrite (1) as

$$[-8\Delta\phi + R\phi]\phi^{-5} = |\sigma|^2\phi^{-12} - \frac{2}{3}\tau^2.$$

The left side is R_h, the scalar curvature of the conformally changed metric. So we have a solution or not depending on Yamabe class of g. For instance, if $\sigma \equiv 0$, the the right hand side is negative, and so can only solve if g has $Y_g < 0$, i.e. g is Yamabe negative class. See fig 7 for which ones have solution, as found in Jim Isenberg's paper from 1995.

Now for the constraint problem, we get to specify 1. a conformal class, 2. a conformal momentum and 3. a constant mean curvature.

The solution technique is the method of sub and supersolutions: We want to solve $-8\Delta\phi + R\phi - |\sigma|^2\phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 0$. We say ϕ_+ is a supersolution of this equation if it satisfies it but ≥ 0 rather than $= 0$, and ϕ_- is a subsolution if the same holds, but with ≤ 0. If ϕ_+, ϕ_- are super/subsolutions, with $0 < \phi_- \leq \phi_+$, then there exists a solution ϕ in the middle, $\phi_- \leq \phi \leq \phi_+$. And so the game becomes to just find sub and supersolutions.