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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: James Dilts Email/Phone: JDilts@uoregon.edu

Speaker's Name: David Maxwell

Talk Title: The conformal method of constructing Cauchy data

Date: 9/10/13 Time: 11:00 am / pm (circle one)

List 6-12 key words for the talk: Cauchy data, conformal method, near-CMC, far-from-CMC

Please summarize the lecture in 5 or fewer sentances: He continued talking of the conformal method, in the non-CMC regime. He talked of current problems, and results in both the positive and negative directions. He also gave 3 open problems.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Maxwell - 2nd Talk

Fig 1

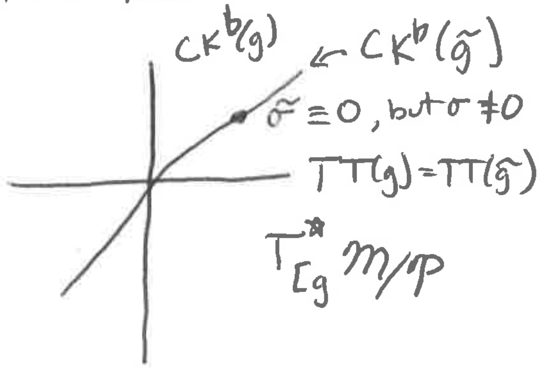

Fig 7

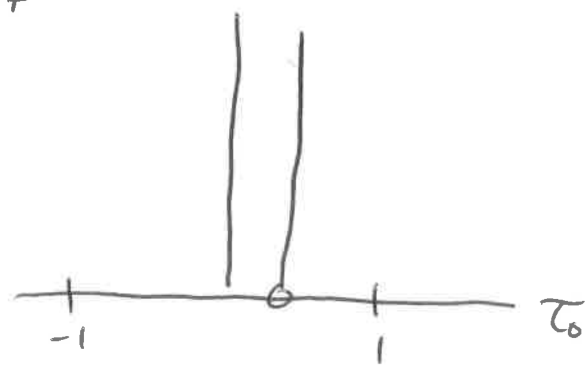

Fig 2

	−	0	+
$\sigma \not\equiv 0, \tau \not\equiv 0$	✓*	✓	✓
$\sigma \equiv 0, \tau \not\equiv 0$	✓*	X	X

Fig 8

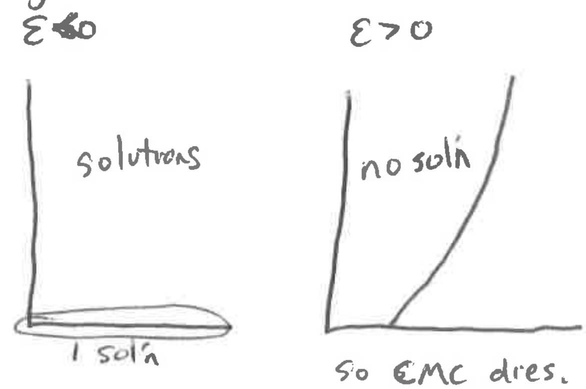

Fig 3

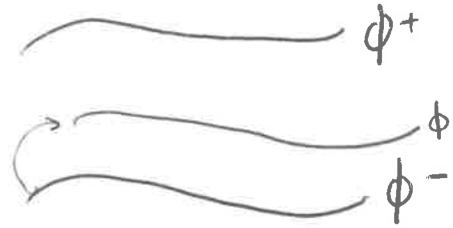

Fig 4

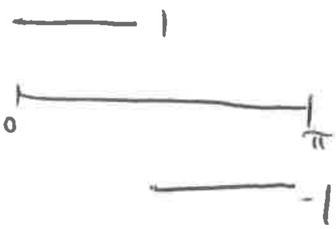

Fig 9

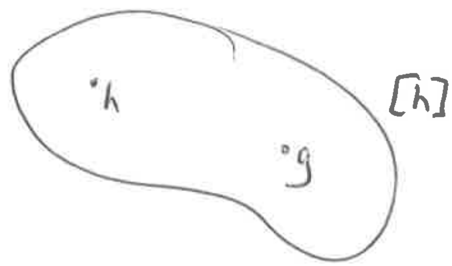

Fig 5

Fig 6

THE CONFORMAL METHOD OF CONSTRUCTING CAUCHY DATA FOR THE EINSTEIN EQUATIONS

DAVID MAXWELL

This is a continuation of previous lecture, so we will assume our slice M^3 is compact and vacuum again.

See the chart from the previous talk, the last figure, that contains the chart for solvability of Lichnerowicz equation when we have constant mean curvature τ. This gives a complete parametrization of constant mean curvature (CMC) data on compact manifolds. But what about non-CMC solutions?

Due to York: '73. First demonstrates : $C^\infty(S_2(M)) = Trace(g) \oplus [CK(g) \oplus TT(g)]$, where $CK(g)$ is the conformal Killing operator of some vector field, i.e. $LW_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{3} g_{ij} \nabla_k W^k$ of some vector field W. $TT(g)$ is the transverse traceless symmetric 2-tensors.

Prescription:

(1) Specify free data (g_{ab}, A_{ab}, τ), where A is a symmetric and trace free 2-tensor.
(2) Solve $\nabla^a LX_{ab} = \nabla^a A_{ab}$, and then let $\sigma_{ab} = A_{ab} - LX_{ab}$, which is then TT.
(3) Seek data (i.e. a solution to the constraints) of the form $h_{ab} = \phi^4 g_{ab}$ and $K_{ab} = \phi^{-2}[\sigma_{ab} + LW_{ab}] + \frac{\tau}{3}\phi^4 g_{ab}$. This implies we need to solve

$$-8\Delta\phi + R\phi - |\sigma + LW|^2\phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 0$$

$$\text{div} LW = \frac{2}{3}\phi^6 d\tau$$

We will call these the LCBY equations, for the discoverers [Lichnerowicz, Choquet-Bruhat, York], where ϕ is an unknown positive function and W is an unknown vector field.

Central question: Is the conclusive success of the CMC conformal method realized by this (or perhaps some related) system for non-CMC initial data?

Clearly, if $d\tau = 0$, this system reduces to the other we constructed. We've had only limited success solving this system so far.

Virtues:

(1) Given a background metric g and a solution (h, K) of the Einstein constraint equations with $h \in [g]$, there exists unique (g, σ, τ) leading to (h, K). (We will just jump to the chase by just specifying σ instead of A.)

(2) The solvability of each individual equation is well understood. For the momentum constraint, $\mathrm{div} LW = \frac{2}{3}\phi^6 d\tau$ with given ϕ, is solvable whenever $\phi^6 d\tau$ is L^2 orthogonal to the set of conformal Killing fields (CKFs), i.e. solutions of $LX = 0$. Typically, [generically even,] there are none, so this equation would always be solvable. For the other equation, the Lichnerowicz equation, we write it as

$$-8\Delta\phi + R\phi = |S|^2\phi^2 - \frac{2}{3}\tau^2\phi^5$$

Then by Maxwell '05, a solution exists if one of the following holds:
(a) $Y_g > 0$ and $S \not\equiv 0$,
(b) $Y_g = 0$, $S \not\equiv 0$ and $\tau \not\equiv 0$
(c) $Y_g < 0$, there exists $\bar{g} \in [g]$ with $R_{\bar{g}} = -\tau^2$. d) $Y_g = 0$, $S \equiv 0$, $\tau \equiv 0$.
This reduces solvability to prescribed scalar curvature, which in this circumstance was solved in '95 by Rauzy. We just need the zero set of τ to be not too big. We'll call such τ for c) Yamabe negative admissible, or YNA for short.

If we let $S = \sigma + LW$, then $S \equiv 0$ implies that $\sigma \equiv 0$ and τ constant.

Vices:

(1) If there are CKFs, we can't always solve the conformal momentum constraint. Little is known if there are CKFs.
(2) There is a lack of conformal covariance. With the CMC method, if we are given (g, σ, τ), and compare with a different background metric in the same conformal class, $(\tilde{g}, \tilde{\sigma}, \tilde{\tau})$, where the $\tilde{\sigma}$ and $\tilde{\tau}$ are appropriately conformally transformed, you still get *same* solution (h, K) if you solve the Lichnerowicz equation. However, in non-CMC case, this doesn't work.

Let $K_{ab} = \phi^{-2}[\sigma_{ab} + LW_{ab}] + trace$. The ϕ^{-2} means transforming like covector, which σ is, but LW transforms like a vector. The identification of vectors with covectors depends on the choice background metric. See Fig 1. And so we get different parameterizations of the covector space, and so different parameterizations via the conformal method.

So what is right generalization of the table from the beginning? The new table would be expected to be as in Fig 2, where * means that there is a solution as long as τ is YNA. This table exactly is correct for near-CMC data. By Isenberg, O'Murchada in '04, for $Y_g \geq 0$, $\sigma \equiv 0$ and $\tau = \tau_0 + \xi$ where τ_0 constant, but ξ is not, if $|\tau_0|$ is large enough (compared to the other given data) there is no solution. This gives the x's in the table.

There is related work by Rendell. He gave data on $S^1 \times S^2$ with the standard metric (so $Y_g > 0$) and with $\sigma \equiv 0$. For this data, there is either no solution or multiple solutions. This is at least consistent with our desired chart.

As far as existence goes for near-CMC, the general strategy is to start with some positive ϕ, solve the momentum equation, get a W, then use that to solve Lichnerowicz equation to find a new ϕ. We call this combined map $\eta_{\sigma,\tau}$. Solutions to LCBY equations then correspond to fixed points of $\eta_{\sigma,\tau}$. We have many of these results now. For instance, if $|\nabla\tau|/|\tau|$ is small and $|\nabla\tau|$ is small, then $\eta_{\sigma,\tau}$ is a contraction map (due to Isenberg and Moncrief in '96 for $Y_g < 0$, and to Isenberg, Clausen and Allen for $Y_g \geq 0$)

An alternative approach is the Leray-Schauder fixed point theorem. Global barriers: ϕ_+ is a global supersolution if whenever $\phi \leq \phi_+$ and W is the solution of the momentum constraint for ϕ, then ϕ_+ is a supersolution for the Lichnerowicz equation for W, as above. We have a similar idea for global subsolution.

If ϕ_+, ϕ_- are global sub and super solutions, we get a fixed point of the equation by Leray-Schauder stuck between the two, but no uniqueness. This was used by Holst, Nagy, Tsogtgerel to find the first far-from-CMC data. If $Y_g > 0$, τ is arbitrary, σ is small but $\not\equiv 0$ and matter is present ($\rho \not\equiv 0$) but not too big, then there exists a solution to the LCBY equations. The key part is new global supersolution: we can find conformal factor ψ such that $\psi^4 g = \hat{g}$ such that $R_{\hat{g}} > 0$. We then take $\epsilon\psi$ for ϵ small depending on τ, and then take σ small depending on ϵ. Then $\epsilon\psi = \phi^+$ is a global supersolution.

His [Maxwell's] contribution was to get rid of the matter requirement by finding a subsolution as long as there is a global supersolution already.

Theorem 0.1 (Maxwell 09?). *Let (M, g) be compact with no CKFs and one of following true:*

(1) $Y_g > 0$, $\sigma \not\equiv 0$,
(2) $Y_g \neq 0$, $\sigma \not\equiv 0$, $\tau \not\equiv 0$,
(3) $Y_g < 0$, τ YNA.

Then if ϕ_+ is a global supersolution, then there exists a solution of the LCBY equations. Uniqueness is not known for these solutions.

This is nice because it lines up with the chart (fig 2). It gives us partial confirmation of the $Y_g > 0$ check mark in figure 2.

Fly in the ointment: (Maxwell '09) We consider $S^1 \times T^2$ and take particular data that depends only on S^1 with g the flat background metric, and families of σ of controlled size. We also take $\tau = \tau_0 + \xi$ with ξ as in fig 4. This is somewhat equivalent to a 2nd derivative discontinuity for the slice we're looking at. Even though it is discontinuous, we could still have a smooth solution of the LCBY equations. See fig 5 for what we would expect to happen, for which data is solvable. Instead we get fig 6. The chart only gives the number of solutions with the same symmetry as the data, so "no solutions" and "one solution" could have more solutions with less symmetry. This is like the Yamabe-null case of the HNT [Holst, Nagy, Tsogtgerel] solution. If we pick τ_0, then pick small enough σ, we get a solution. However, we would hope that the small σ condition was just a temporary thing, rather than a permanent feature.

There are similar results on $S^1 \times S^2$ (so $Y_g > 0$) by Cheng, student of Tsogtgerel, and so the weirdness continues for Yamabe positive.

What if I picked different metric in same conformal class for $S^1 \times T^2$? See fig 7. The homothetic family family moves. This means that a different background metric says that some other mean curvature is special, since the homothetic family moves along the axis.

Fragility for same symmetric data: We consider $\operatorname{div} K - (1+\epsilon) d\tau = 0$ instead for momentum constraint. See fig 8 for a similar chart for this case. Thus the exact value of the constant coefficient of $d\tau$ matters, not just the sign or something as we might have hoped.

This is bad news for parametrization, since all this messiness suggests it won't work, at least nicely. We would need a lot of clarification.

Open problems:

(1) Does the naive picture (fig 2) hold generically? (i.e. without CKFs, for instance.) This could still happen because $S^1 \times T^2$ has lots of CKFs. To approach this, is $\sigma \equiv 0$ impossible if $Y_g \geq 0$? a) yes, of course if CMC or near-CMC. b) Some evidence for yes in general with Rendell's example. c) Mild counterexamples from fig 6. The hard part is that $\sigma \equiv 0$ depends on the choice of background metric, i.e. from (h, K) you can't tell if $\sigma \equiv 0$ is true, you must have g as well.

(2) What about other parameterizations? For CMC, this method works great, but for non-CMC it feels ad hoc. There are other contenders: conformal thin sandwich method, but the theory is the same as the standard conformal method. There's also "method B." The idea is that we're normally decomposing with respect to g instead of h, so if we do it with respect to to h, we get an ugly $|LW|\phi^5$ with a bad sign somewhere.

However, if we take the conformal constraints [LCBY equations], if we have a solution, then LW is on the order of ϕ^6. If we put that into the Lichnerowicz, then we get a term like $|LW|\phi^5$, which is with the bad sign. Thus the difficulty is still there in the LCBY equations, it's just hidden in the coupling.

(3) What's the deal with CKFs? a) They play no role in the CMC theory. b) No role in Near-CMC non-existence. c) All* non-CMC existence proofs require no CKFs (or perhaps at least $X(\tau) = 0$ for any CKF, which is often enough to show $\tau \equiv 0$.)

Suppose I have a solution (g_{ab}, K_{ab}) and X^b is a CKF. Take the momentum constraint $\nabla^a K_{ab} - \nabla_b \tau = 0$, multiply it by X, integrate over M, then integrate by parts. We find we need $\int X(\tau) dV_h = 0$ or $\int X(\tau)\phi^6 dV_g = 0$. But we don't even know what h or g *is* until we solve using the τ, so this is nigh impossible to guarantee if there are CKFs. We might hope it'd work anyway, but Maxwell in '11 showed that there exists CMC data on S^3 and arbitrarily small perturbations τ_ϵ of τ such that $(g_\sigma, \tau_\epsilon)$ is conformal data with no solution.