Ancient Solutions to Geometric Flows

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We will discuss ancient or eternal solutions to geometric parabolic partial differential equations.

These are special solutions which exist for all time

\[-\infty < t < T \quad \text{where} \quad T \in (-\infty, +\infty].\]

They appear as blow up limits near a singularity.

Understanding ancient and eternal solutions often sheds new insight to the singularity analysis
In this talk we will address:

- the *classification* of *ancient* solutions to *parabolic* partial differential equations, with emphasis to *geometric flows*: *Mean Curvature* flow, *Ricci* flow and *Yamabe* flow.

- methods of *constructing* new ancient solutions from the *gluing* of two or more *solitons* (self-similar solutions).

- new techniques and future research directions.
**Definition:** A solution $u(\cdot, t)$ to a parabolic equation is called **ancient** if it is defined for all time $-\infty < t < T$, $T < +\infty$.

Ancient solutions typically arise as blow up limits at a **type I** singularity.

**Definition:** A solution $u(\cdot, t)$ to a parabolic equation is called **eternal** if it is defined for all $-\infty < t < +\infty$.

Eternal solutions as blow up limits at a **type II** singularity.
Solitons (self-similar solutions) are typical examples of ancient or eternal solutions and often models of singularities.

Some typical examples of solitons to geometric PDE are:

- **Spheres:**

- **Cylinders:**

- **Translating solitons:**
A well known technique introduced by R. Hamilton (1995) has been widely used to characterize as solitons the eternal solutions to geometric flows which attain a space-time curvature maximum.

Such solutions typically appear as carefully chosen blow up limits near type II singularities.

Its proof relies on a clever combination of the strong maximum principle and Li-Yau type differentiable Harnack estimates.
However, there exist other ancient or eternal solutions which are not solitons.

These, often may be visualized as obtained from the gluing as $t \to -\infty$ of two or more solitons.

Obtaining more information about such solutions, often leads to the better understanding of the singularities.

**Objective:** How to construct such solutions and how to characterize them among all ancient or eternal solutions.
Goal: Characterize all ancient or eternal solutions to a geometric flow under natural geometric conditions:

- Being a soliton (self-similar solution)
- Satisfying an appropriate curvature bound as $t \to -\infty$:
  - Type I: global curvature bound after typical scaling.
  - Type II: solutions which are not type I.
- Satisfying a non-collapsing condition.
Liouville’s theorem for the heat equation on manifolds

- Let $M^n$ be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}(M^n) \geq 0$.

- **Yau (1975):** Any positive harmonic function $u$ on $M^n$ must be constant.

- This is the analogue of Liouville’s Theorem for harmonic functions on $\mathbb{R}^n$.

- **Question:** Does the analogue of Yau’s theorem hold for positive solutions of the heat equation $u_t = \Delta u$ on $M^n$?

- **Answer:** No. Example $u(x, t) = e^{x_1 + t}$, $x = (x_1, \ldots, x_n)$ on $M^n := \mathbb{R}^n$. 
A Liouville type theorem for the heat equation

- **Souplet - Zhang (2006):** Let $M^n$ be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}(M^n) \geq 0$.

  (a) If $u$ be a positive ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that

  
  $$u(p, t) = e^{o(d(p) + \sqrt{|t|})} \quad \text{as } d(p) \to \infty$$

  then $u$ is a constant.

  (b) If $u$ be an ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that

  
  $$u(p, t) = o(d(p) + \sqrt{|t|}) \quad \text{as } d(p) \to \infty$$

  then $u$ is a constant.

- **Proof:** By using a local gradient estimate of Li-Yau type on large appropriately scaled parabolic cylinders.
Consider positive solutions $u > 0$ of the semilinear elliptic equation
\[ \Delta u + f(u) = 0, \quad \text{on } \mathbb{R}^n. \]

Well known example related to the Yamabe problem is $f(u) = u^{\frac{n+2}{n-2}}$.

Gidas, Ni and Nirenberg (1979): Solutions $u > 0$ with mild decay condition as $|x| \to +\infty$ are rotationally symmetric.

Many related important subsequent results including those by Cafarelli, Gigas and Spruck and Berestycki and Nirenberg.
Consider next the **semilinear heat equation**

\[
(\star_{SL}) \quad u_t = \Delta u + u^p \quad \text{on } \mathbb{R}^n \times (0, T)
\]

in the subcritical range of exponents \(1 < p < \frac{n+2}{n-2}\).

It provides a prototype for the **blow up analysis** of geometric flows.

In particular in **neckpinches** of solutions to the Ricci flow and Mean Curvature flow.

Also in the characterization of **rescaled limits** as \(t \to -\infty\) of ancient solutions.
The rescaled semi-linear heat equation

- Self-similar scaling at a singularity at \((a, T)\):

\[
    w(y, \tau) = (T - t)^{\frac{1}{p-1}} u(x, t), \quad y = \frac{x - a}{\sqrt{T - t}}, \quad \tau = -\log(T - t).
\]

- Giga - Kohn (1985): \(\|w(\tau)\|_{L^\infty(\mathbb{R}^n)} \leq C, \tau > -\log T\).

- The rescaled solution satisfies the equation

\[
    w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p - 1} + w^p.
\]

- Objective: To analyze the blow up behavior of \(u\) one needs to understand the long time behavior of \(w\) as \(\tau \to +\infty\).

- This is closely related to the classification of bounded eternal solutions of (*).
**Problem:** Provide the classification of bounded positive eternal solutions $w$ of equation

\[(\star) \quad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.\]

- **Eternal** means that $w(\cdot, \tau)$ is defined for $\tau \in (-\infty, +\infty)$.
- The only **steady states** of $(\star)$ are the **constants**:

  \[w = 0 \quad \text{or} \quad w = \kappa, \quad \text{with} \quad \kappa := (p - 1)^{-\frac{1}{p-1}}.\]

- **Theorem (Giga - Kohn '87)** \(\lim_{\tau \to \pm\infty} w(\cdot, \tau) = \text{steady state.}\)
- **Space independent eternal solutions**:

  \[\phi(\tau) = \kappa(1 + e^\tau)^{-\frac{1}{(p-1)}}.\]
Classification of Eternal solutions

- **Theorem (Giga - Kohn ’87 and Merle - Zaag ’98):** If $w$ is bounded positive eternal solution of $(\star)$ defined on $\mathbb{R}^n \times (-\infty, +\infty)$, then

  $$w = 0 \text{ or } w = \kappa \text{ or } w = \phi(\tau - \tau_0).$$

- **Main difficulty (Merle - Zaag):** Classify the orbits $w(\cdot, \tau)$ that connect the two steady states:

  $$\lim_{\tau \to -\infty} w(\cdot, \tau) = \kappa \text{ and } \lim_{\tau \to +\infty} w(\cdot, \tau) = 0$$

- **Important Liouville type results** related to equation $(\star_{SL})$ by: P. Polacik, P. Quittner, T. Bartsch, P. Souplet, E. Yanagida among others.
Although Liouville type results often appear with respect to elliptic equations, there are not many such results available in the parabolic setting.

G. Koch, N. Nadirashvili, G. Seregin and V. Sverak (2009):
(i) Liouville type result for ancient bounded solutions $u(x, t)$ of the 2-dim Navier Stokes equations.

(ii) Also, similar result for bounded, axi-symmetric with no swirl solutions $u(x, t)$ of the 3-dim Navier Stokes equations.
Let $\Gamma_t$ be a family of closed curves which is an embedded solution to the Curve shortening flow, i.e. the embedding $F : \Gamma_t \rightarrow \mathbb{R}^2$ satisfies

$$\frac{\partial F}{\partial t} = -\kappa \nu$$

with $\kappa$ the curvature of the curve and $\nu$ the outer normal.

Gage (1984); Gage and Hamilton (1996); Grayson (1987): the CSF shrinks $\Gamma_t$ to a round point.

Problem: Classify the ancient compact embedded solutions to the Curve shortening flow.
Ancient Convex solutions to the CSF

- The curvature $\kappa$ of $\Gamma_t$ evolves, in terms of its arc-length $s$, by
  $$\kappa_t = \kappa_{ss} + \kappa^3.$$  

- **Definition**: $\Gamma_t$ is type I if $\limsup_{t \to -\infty} \sqrt{|t|} \max_{\Gamma_t} \kappa(\cdot, t) < \infty$. Otherwise, $\Gamma_t$ is of type II.

- **Type I solution**: the contracting circles.

- **Type II solution**: the Angenent ovals (paper clips). These are ancient convex solutions in closed form which are not solitons.
The Angenent ovals (paper clips) as \( t \to -\infty \) may be visualized as two grim reaper solutions glued together.

**Theorem (D., Hamilton, Sesum - 2010)**
The only ancient convex solutions to the CSF are the contracting spheres or the Angenent ovals.

**Proof:** It is based on various monotonicity formulas and the fact that at its singular time any solution becomes circular with very sharp rates of convergence.
**Question:** Do they exist **non convex compact** embedded solutions to the curve shortening flow?

**Angenent (2011):** Presents a *YouTube video* of an ancient solution to the CSF built out from one Yin-Yang spiral and one Grim Reaper.

**S. Angenent:** is currently working on a rigorous construction of these solutions.
Let $M_t, \ t \in (-\infty, T)$ be a smooth ancient compact solution of the Mean curvature flow

$\frac{\partial F}{\partial t} = -H \nu$

$H(p, t)$ is the Mean curvature and $\nu$ a choice of unit normal.

**Problem:** Understand ancient compact solutions $M_t$ of the Mean curvature flow.
Ancient non-collapsed solutions to MCF

- Weimin Sheng and Xu-Jia Wang; Ben Andrews: Introduced an \( \alpha \)-noncollapsed condition.

\[
B = B_{\alpha} \frac{1}{H(p)}
\]

- Haslhofer & Kleiner (2013): Ancient compact + \( \alpha \)-noncollapsed MCF solution \( \Rightarrow \) convex.

- Convex compact + self-similar MCF solution \( \Rightarrow \) sphere.

- Ancient ovals: Any compact and \( \alpha \)-noncollapsed solution to MCF which is not self-similar.

- Other ancient solutions to MCF: compact and collapsed.
Problem: Provide the classification of all Ancient ovals.

B. White (2003): Existence of certain Ancient ovals with $O(k) \times O(l)$ symmetry. We call them White ancient ovals.

Haslhofer & Hershkovits (2013): Give more details in the existence proof of the White Ancient ovals.

Angenent (2012): establishes the formal matched asymptotics of all Ancient ovals as $t \to -\infty$.

They are small perturbations of ellipsoids.
S. Angenent, D., and N. Sesum (2015): All ancient ovals with $O(1) \times O(n)$ symmetry have unique asymptotics as $t \to -\infty$, and satisfy Angenent’s precise matched asymptotics:

Geometric properties $t \to -\infty$: type II ancient solutions

$$\text{diam}(t) \approx \sqrt{8|t| \log|t|} \quad \text{and} \quad H_{\text{max}}(t) \approx \frac{\sqrt{\log|t|}}{\sqrt{2|t|}}.$$ 

The proof involves: the analysis of the linearized operator at the cylinder, Huisken’s monotonicity formula and carefully constructed barriers at the intermediate region.
Uniqueness of Ancient MCF ovals

- **Work in progress:** to establish such asymptotics in the non-symmetric case.

- **Next Step:** Establish the uniqueness of the Ancient ovals.

- **Conjecture 1:** The Ancient ovals with $O(l) \times O(k)$ symmetry are uniquely determined by their asymptotics at $t \to -\infty$.

- **Hence:** they are unique (up to dilation and translation in rescaled time).

- **Conjecture 2:** All Ancient ovals are $O(l) \times O(k)$ symmetric.
Consider an **ancient solution** of the **Ricci flow**

\[
(RF) \quad \frac{\partial g_{ij}}{\partial t} = -2 R_{ij}
\]

on a compact manifold \( M^2 \) which exists for all time \(-\infty < t < T\) and becomes singular at time \( T \).

In dim 2, we have \( R_{ij} = \frac{1}{2} R g_{ij} \), where \( R \) is the scalar curvature.

**Hamilton (1988), Chow (1991):** After re-normalization, the metric becomes **spherical** at \( t = T \).

**Problem:** Provide the **classification** of all ancient compact solutions.
Choose a parametrization $g_{S^2} = d\psi^2 + \cos^2\psi\, d\theta^2$ of the limiting spherical metric.

We parametrize our solution as $g(\cdot, t) = u(\cdot, t)\, g_{S^2}$.

Then the (RF) becomes equivalent to the fast-diffusion equation:

$$u_t = \Delta_{S^2} \log u - 2, \quad \text{on } S^2 \times (\mathbb{R}, T).$$

Provide the classification of all ancient solutions.
Examples of compact solutions on $S^2$

- **Type I** solution: the contracting spheres.

- **Type II** solution: the King-Rosenau solution of the form:

  \[ u(\psi, t) = \left[ a(t) + b(t) \sin^2 \psi \right]^{-1}, \quad t < T. \]

  As $t \to -\infty$ the King-Rosenau solution looks like two cigar solitons glued together.
The classification result

**Theorem:** (D., Hamilton, Sesum - 2012)
The only ancient solutions to the Ricci flow on $S^2$ are the contracting spheres and the King-Rosenau solutions.

**Proof:** combines geometric arguments and PDE techniques.

i. a monotonicity formula and uniform a priori $C^{1,\alpha}$ estimates that allow us to pass to the limit as $t \to -\infty$.

ii. geometric arguments that allow us to classify the backward limit as $t \to -\infty$.

iii. maximum principle arguments that allow us to characterize the King-Rosenau solutions among type II solutions.

iv. an isoperimetric inequality that allows us to characterize the contracting spheres among type I solutions.
The characterization of King solutions

- To capture the King solutions we consider the scaling invariant nonotone quantity

\[ Q(x, y, t) := \bar{v} \left[ (\bar{v}_{xxx} - 3\bar{v}_{xxy})^2 + (\bar{v}_{yyy} - 3\bar{v}_{xxy})^2 \right] \]

where \( \bar{v} := \bar{u}^{-1} \) is the pressure in plane coordinates.

- Using complex variable notation \( z = x + iy \), this quantity is nothing but

\[ Q = \bar{v} |\bar{v}_{zzz}|^2. \]

- The quantity \( Q \) is well defined.

- It turns out that \( Q \equiv 0 \) implies that \( \bar{v} \) is one of the King solutions.

- To establish that \( Q \equiv 0 \) we prove that:

  i. \( Q_{\text{max}}(t) \) is decreasing in \( t \) (by considering its evolution equation), and

  ii. \( \lim_{t \to -\infty} Q_{\text{max}}(t) = 0. \)
3-dimensional Ricci flow: The analogue of the 2-dimensional King-Rosenau solutions have been shown to exist by G. Perelman. They are not given in closed form, they are type II and k-noncollapsed.

Other collapsed compact solutions in closed form have been found by V.A. Fateev in a paper dated back to 1996.

Conjecture: The only k-noncollapsed ancient and compact solutions to the 3-dimensional Ricci flow are the contracting spheres and the Perelman solutions.

Brendle, Huisken & Sinestrari (2011): Present a pinching curvature condition that characterizes the ancient compact solutions to the 3-dimensional Ricci as contracting spheres.
We will conclude by discussing ancient solutions $g = g_{ij}$ of the Yamabe flow on $S^n$, $n \geq 3$.

The Yamabe flow may be viewed as the higher dimensional analogue of the 2-dim Ricci flow.

It is the evolution of metric $g(\cdot, t)$ conformally equivalent to the standard metric on $S^n$ by

$$\frac{\partial g}{\partial t} = -Rg \quad \text{on} \quad -\infty < t < T$$

where $R$ denotes the scalar curvature of $g$.

**Question:** Is it possible to provide the classification of all such ancient solutions?
The Yamabe flow - Background

- Let \((M^n, g_0)\), \(n \geq 3\) be a compact manifold without boundary.
  The scalar curvature \(R\) of a metric \(g = v^{\frac{4}{n-2}} g_0\) conformal to \(g_0\) is given by
  \[
  R = -v^{-\frac{n+2}{n-2}} \left( c_n \Delta_{g_0} v - R_0 v \right)
  \]
  where \(R_0\) denotes the scalar curvature of \(g_0\).


- S. Brendle (2007): convergence of the normalized flow to a metric of constant scalar curvature (up to a mild technical assumption for \(\text{dim } n \geq 6\)).

- Previous important works: Hamilton '89, Chow '92, Ye '94, Schwetlick-Struwe '2003.
Let $g = \nu^{\frac{4}{n-2}} g_{S^n}$ be an ancient solution to the Yamabe flow, which is conformal to the standard metric on $S^n$.

The function $\nu$ evolves by the fast diffusion equation

$$(\nu^{\frac{n+2}{n-2}})_t = \Delta_{S^n} \nu - c_n \nu \quad \text{on } S^n \times (-\infty, T).$$

Let $g = \bar{\nu}^{\frac{4}{n-2}} g_{\mathbb{R}^n}$ after stereographic projection. Then,

$$(\bar{\nu}^{\frac{n+2}{n-2}})_t = \Delta \bar{\nu} \quad \text{on } \mathbb{R}^n \times (-\infty, T).$$

**Definition:** An ancient solution is called of type I if:

$$\limsup_{t \to -\infty} (|t| \max_{S^n} |Rm| (\cdot, t)) < \infty.$$ Otherwise, it is called of type II.
The King Solutions

- **J.King (1993):** discovered non-self similar type I ancient compact solutions to the (YF) on $S^n$ in closed form.

- **King solutions:** $g = \hat{v}_K(\cdot, t) \frac{4}{n-2} g_{\mathbb{R}^n}$, where

  $$\hat{v}_K(x, t) = (a(t) + 2b(t)|x|^2 + a(t)|x|^4)^{-\frac{n-2}{4}}, \quad x \in \mathbb{R}^n.$$ 

- As $t \to -\infty$ they converge (after rescaling) to two Barenblatt type self-similar solutions (shrinking solitons) joined by a long cylindrical neck.

![Cylindrical neck diagram](image)
Question 1: Are the contracting spheres and the King solutions the only examples of type I ancient solutions?

Question 2: Are there any type II ancient solutions?
New Type I solutions to the Yamabe flow

- Recent work: (D., del Pino, J. King and N. Sesum - 2015)
  There exist infinite many other type I ancient solutions.

- As $t \to -\infty$ they look as two self-similar solutions $v_\lambda, v_\mu$
  connected by a cylinder and moving with speeds $\lambda > 0, \mu > 0$.

- Our solutions are not given in closed form but we show very sharp asymptotics.

- In similar spirit to the work by Hamel and Nadirashvili (1999)
  where they construct ancient solutions for the KPP equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}.$$
We look for rotationally symmetric shrinking solitons of the (YF) expressed in cylindrical coordinates $g = v^{\frac{4}{n-2}} g_{cyl}$. $v(x, \tau)$ satisfies (after a type I rescaling) the equation:

\[ (\ast) \quad (v^\frac{n+2}{n-2})_\tau = v_{xx} - v + v^\frac{n+2}{n-2}. \]

Shrinking solitons (or traveling waves): $\forall \lambda \geq 1$ there exist a solution $v_\lambda = V_\lambda(x - \lambda \tau)$ of $(\ast)$ with cylindrical behavior:

\[ V_\lambda(x) \approx 1 - C_\lambda e^{-\gamma_\lambda x}, \quad \text{as } x \to +\infty. \]

Theorem: (D., J. King and N. Sesum)

$L^1$ stability of the traveling wave solutions $v_\lambda$. 

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Ancient Solutions to Geometric Flows
Shrinking solitons with cylindrical behavior

- Consider **shrinking solitons** in cylindrical coordinates and after a type I scaling.

- Traveling wave to the right: \( v_{\lambda,h} = V_\lambda(x - \lambda \tau + h) \)

- Traveling wave to the left: \( \bar{v}_{\mu,h'} = V_\mu(-x - \mu \tau + h') \)

- Cylinder: \( \xi_k(\tau) \approx 1 - k \, e^{\tau/2} \), as \( \tau \to -\infty \).
Theorem: (D., del Pino, J. King and N. Sesum)

There exist a five parameter family $v_{\lambda,\mu,h,h',k}$ of type I ancient solutions of the Yamabe flow on $S^n \times (-\infty, T)$.

In terms of the pressure function $f := v^q$, $q := -\frac{4}{n-2}$ it satisfies:

$$v_{\lambda,\mu,h,h',k}^q \approx v_{\lambda,h}(x, \tau) + \xi_k(\tau)^q + \bar{v}_{\mu,h'}(x, \tau).$$

Proof: By the construction of precise ancient barriers.
Question: Are there any type II ancient solutions to (YF)?

D., del Pino and Sesum (2013):
We construct a class of ancient solutions of the Yamabe flow on $S^n$ which (after re-normalization) converge as $t \to -\infty$ to a tower of n-spheres. They are rotationally symmetric.

The curvature operator in these solutions changes sign and they are of type II.

Our construction also holds for any number of bubbles.
Our construction may be viewed as a **parabolic analogue** of the **elliptic gluing** technique.

**Elliptic gluing**: pioneering works by Kapouleas ’90 -’95 and by Mazzeo, Pacard, Pollack, Ulhenbeck among many others.

**Brendle & Kapouleas (2014)**: construct new **ancient compact solutions** to the **4-dim Ricci flow** by parabolic gluing.

**Future research direction**: apply parabolic gluing on other geometric flows.
We discussed ancient solutions to geometric parabolic PDE.

Typical examples are either solitons or other special solutions obtained from the gluing as $t \to -\infty$ of solitons.

The only existing classification results heavily rely on knowing the exact form of these ancient solutions.

**Future research direction**: develop new techniques that allow us to characterize and construct other types of ancient or eternal solutions.