Guofang Wei – Local Isoperimetric Constant Estimate for Integral Ricci Curvature.

I. Why and what's integral curvature?

Most often

L^p see curvature $\quad \|Rm\|_p = \left(\int_M |Rm|^p\right)^{1/p} < C$

weaker condition $\quad p = \frac{n}{2}$, scale invariant
(manifold is n-dim)

It occurs Naturally

M^2 Gauss Bonnet ↙ sectional curvature

$$\chi(M) = \frac{1}{2\pi}\int K \, dvdg$$

variation problems
isospectral problem

Keller-Rich flow?

Recent work: (Tim – Z. Zhang) (KRF)

$\|Ric\|_4 \le C$ (in all dimensions)

Bamler – Q Zhang RF dim 4 $\|Ric\|_4 < C$
M Simon
if s is bounded, finite time

Cheeger – Naber $|Ric| \le H$ $|Vol| \ge v$ $Dim \le D$
$\Rightarrow \|Rm\|_q \le C \quad 0 \le q < 2$

Integral Ricci curvature lower bound

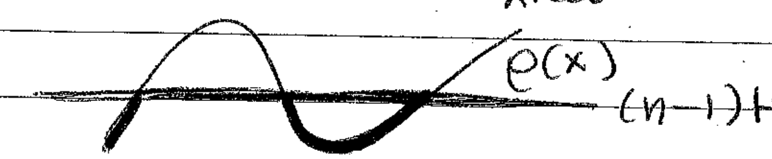
$Ric \ge (n-1)H \rightsquigarrow$ integral version?

$$\|Ric^H\|_{p,R} = \sup_{x \in M} \left(\int_{B(x,R)} \left((n-1)H - \rho(x)\right)_+^p\right)^{1/p}$$

in L^p sense

part of it below $(n-1)H$ Ricci

smallest eigenvalue of Ric.



$Ric \ge (n-1)H \longleftrightarrow \|Ric^H\|_p \equiv 0$

For this talk we will let $H = 0$. Consider the scale invariant geometry.

Notation: $\displaystyle ⨍_{B(x,R)} = \frac{1}{Vol(B(x,R))}\int_{B(x,R)}$

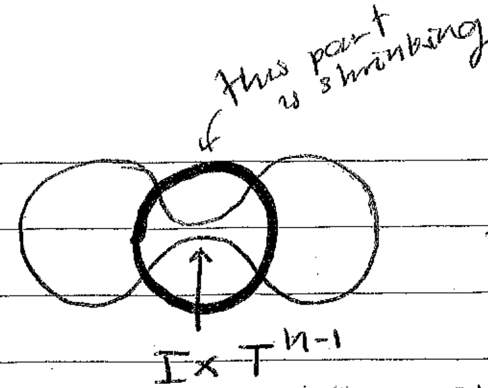
$$R^2\left(⨍_{B(x,R)} (\rho_-)^p\right)^{1/p} = k(x,p,R).$$

So that it is scale invariant

Problem: extends results for pointwise Ricci curvature ⟶ integral curvature

Short answer: Sometimes no and sometimes yes.

No.



vol $\geq V$ always
$\mathrm{Dim} \leq D$
$|k| \leq H$

$$dr^2 + (\epsilon + r)^{2k} d\theta^2$$

$\epsilon \to 0$, $\|Rm\|_p < \infty$ for all $p < \infty$.

Volume doubling doesn't hold

$$\frac{\mathrm{Vol}(B(x,2R))}{\mathrm{Vol}(B(x,R))} \leq C \quad \forall R$$

Yes II. Previous Work (we will only cover what is needed for more recent work).

For $Ric \geq (n-1)H$ a very important tool is the laplacian comparison.
Let $r(x) = d(x, x_0)$, the distance function
$$\Delta r = \Delta_H r$$
Δ_H is the laplacian in the model space
$M^n_H \longrightarrow$ n dimensional, simply connected. $K \equiv H$

When $H = 0$, $\Delta_H r = \frac{n-1}{r}$

$$\psi = \left(\Delta r - \frac{n-1}{r}\right) \longleftrightarrow$$ the part the laplace comparison failed.

If $Ric > 0 \Rightarrow \psi \equiv 0$ usual laplace comparison

THM (Petersen-Wei '97) Given M^n
$p > \frac{n}{2}$, $r > 0$,

$$\|\psi\|_{2p, B(x,r)} \leq C(n,p) \left(\|Ric_-\|_{p, B(x,r)} \right)^{1/2}$$

How much the laplace comparison fails is given by Ric_-.
Laplace comparison in integral.

Volume compressing for integral curvature $R \geq r$

$$\left(\frac{\mathrm{Vol}\, B(x,R)}{R^n}\right)^{1/2p} - \left(\frac{\mathrm{Vol}(B(x,r))}{r^n}\right)^{1/2p} \leq e(n,p)\, R^{1-n/p} \left(\|Ric_-\|_{p, B(x,R)}\right)^{1/2}$$

when $\|Ric_-\|_p = 0$ then it recovers the usual laplacian and volume comparison.

$$\left(\frac{\mathrm{Vol}\, B(x,r)}{\mathrm{Vol}\, B(x,R)}\right)^{1/2p} \geq \left(\frac{r}{R}\right)^{n/2p} \left[1 - e(n,p)\left(k(x,p,R)\right)^{1/2}\right]$$

$$R^2 \left(\fint_{B(x,R)} (Ric_-)^p \right)^{1/p}$$

We have volume doubling when $k(p,R)$ is small. Therefore we assume $k(p,R)$ is small. How do you know when this is true?

When $Vol(B(x,R)) \geq C R^n$

$$\Rightarrow K(x,p,R) \leq C^{-1/p} R^{2-1/p} \Big(\underbrace{\int_{B(x,R)} (Ric_-)^p}_{\text{If this is bounded}} \Big)^{1/p}$$

Note: want $2 - 1/p > 0$

$\Rightarrow k$ is small enough

If $R \leq R_0$ $\|Ric_-\|_p$ is bounded
$\Rightarrow K(p,R)$ is small.

It's optimal in the sense that the result does not hold when $p \leq \frac{n}{2}$

If you have $K(p,R)$ small, then for some r, it controls $K(p,R)$ for all R

$R = 1$.

III New Results.

THM (Dai - Wei - Zhang) M^n $p > n/2$
$\exists\, \epsilon = \epsilon(n,p)$ s.t. if $K(p,1) \leq \epsilon$ then for $x \in M$
$r \leq 1$ $\qquad \sup_{x \in M} \left(⨍_{B(x,1)} |Ric_-|^p \right)^{1/p}$
$\partial B(x,1) \neq \emptyset$

$$\underbrace{Is(B(x,r))}_{\substack{\text{local isometric constant} \\ \text{~~(Dirichlet)~~}}} \leq c(n)\, r \big/ (Vol\, B(x,r))^{1/n}$$

$$Is\, B(x,r) = \sup \left\{ \frac{Vol(\Omega)^{1-1/n}}{Vol(\partial\Omega)} \;\middle|\; \Omega \subset\subset B(x,r) \right\}$$

here $\Omega \cap \partial B(x,r) = \emptyset$

~~COROLLARY~~ REMARK: ① Yang had an earlier estimate but you had to assume $vol(B(x,r)) \geq v > 0$ 1992

② Gallot: had to assume M closed, global isometric constant estimate 1998

COROLLARY: Normalized Sobolev inequality

$$\left(⨍_{B(x,r)} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq c(n)\, r ⨍_{B(x,r)} |\nabla f|$$

$f \in C_0^\infty(B(x,r))$ $\quad r \leq 1$

Volume doubling + Sobolev $\longrightarrow$

Big open question

Gromov: If $⨍ |Rm|^{n/2} \leq \epsilon(n)$
compact $\to M$
$\Rightarrow M$ is almost flat