RIEMANNIAN MANIFOLDS WITH LOWER CURVATURE BOUND

M: complete Riemannian mfd
Sec M: sectional curvature, assume Sec M ≥ k, k ∈ ℝ
Let's consider various geometric ways to interpret this condition (comparison thm)
Sk: Constant curvature k-surface, simply connected
(V) "Hinge" version (angle + geodesic)

\[ d \leq \bar{d} \text{ in } S_k \]

"Hinge" version (geodesic + distance)

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<tr>
<th>d1</th>
<th>d2</th>
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<td>1</td>
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"O" marks corresponding pts.

expressed without angles but with distance

(C) Four point version

- draw corresponding geodesic A's so they have adjacent sides
- constant curvature triangle

condition: sum of angles is ≤ 2π (these are "comparison angles").

No curves or angles necessary, only need metric in this case!

Curv ≥ k!

we reserve "sec" for Riemannian manifolds

Critical Pt Theory for dist functions.

There exists a tangent vector at \( q \) so that no matter which min. geod. you travel on you \( M \), make angle ≥ 2π.

A pt. \( q \) is regular for dist \( p \), all minimal directions from \( q \) to \( p \) are in some cone \( q \) is critical if not regular

\[ \text{ext} \text{nd} \text{ in nbhd} \]

You can construct a local vector field (like Morse theory)
key point: you have an isotopy.

APPLICATIONS (3 to consider)

\[ \text{Thm (Diam Sphere Thm) } (G \times X) \quad \text{see } M \geq 1, \text{ diam } M > D_k \rightarrow \text{ diam of 2-sphere} \]

\[ M \approx S^n \text{ homeomorphic} \]

\[ \text{Thm (Betti Number Thm) (Gromov) } \quad \text{see } M \geq k, \text{ diam } M \leq D_k \]

\[ \text{Note } \begin{array}{l}
(D, k, n) \text{ all real numbers}
\end{array} \]
\[ \dim H^*(M; F) \leq C(n, k, D) \]

Bound m how complicated homology of mfld is.

Thm (Homotopy Finiteness) (G, d, P) Given \( n, k, D \) and \( v_0 \), there are at most finitely many homotopy types of \( M \) with \( \text{sec} \geq k, \text{diam} \leq D, \text{vol} \geq v_0 \).

Consider \( n \geq 1 \), \( \text{diam} M > \frac{\pi}{2} \).

\[ \text{claim} \quad p^* \text{ is unique} \]

\[ \text{dist}(p, p^*) = \text{diam} > \frac{\pi}{2} \text{ only on } p^* \]

For Homotopy Finiteness Thm use Gromov Hausdorff distance for closeness.

Gromov Hausdorff distance \( d_G \).

Given \( Z \): compact metric space. \( A, B \) closed \( \subseteq Z \).

The HAUSDORFF METRIC

\[ d_H^e(A, B) = \inf \{ r : D(A, r) \subseteq B, D(B, r) \subseteq A \} \]

\( D(A, r) \) is the "r-neighborhood".

\[ d_G(X, Y) = \inf_{x, y} d_H^e(x, y) \]

If 2 spaces have distance 0, they are isometric.

Note: it suffices to choose \( Z = X \uplus Y \) (the disjoint union).

Q: what does \( X_n \rightarrow_G X \) mean?

\( GH \) distance goes to 0. Construct metric on disjoint union and look at convergence wrt Hausdorff distance.

\[ \inf \text{ space of 2 pts dist 1 apiece} \]

Example: \( X = \text{pt}, \quad Y = \mathbb{R} \)

\[ d_G(x, y) = \frac{1}{2} \]

\( X \subseteq \text{space of } 2 \)

\[ d_G(x, y) = \frac{1}{2} \]

\[ \text{take disjoint union as before \& definedist.} \]

Any compact metric space is so close to any wanted to a finite metric space, this is a very coarse topology.

Theorem of Gromov \( \Rightarrow \) relatively compact metric spaces

\( X \) class of compact metric spaces is \( GH \) precompact if \( \exists \epsilon > 0 \): Any \( X \in X \) can be covered by \( \leq C(\epsilon) \) \( \epsilon \)-balls. (The number is uniform.)
Let's consider some convergence examples.

Ex

- squash → has top & bottom
- squash
- It's NOT a disk - it's a double disk!
- act by rot
- ... → interval

Now look at not-so-nice examples

- smooth sphere!
- attach more and more handles which get smaller each iteration
- topology changes drastically, curvature $> - \infty$

Preserved properties
- $X_n$ length space
- $\lim X_n = X$ → $\mathbb{R}^n$ mid pt
- length space $\leftrightarrow$ has almost $\varepsilon$-property for all pts

Def: Alexandrov space $X$

1. $X$ length space
2. $\exists \kappa: \text{Curv}\, X \geq \kappa$
3. $\dim X \leq \kappa$

Consequently, $M_n$ has curv $M_n \geq k$ then $M_n$ $\text{Alex}$ (Alexandrov space).

$\forall$ length spaces

Ex $M = X$, sec $M \geq k$, $X = \lim M_n$, sec $M \geq k$

$\Omega_{\text{convex IR}} \subset \Omega$ $\Omega$ Alexandrov. curv $\geq 0$.

Ex $E$: Alexandrov w/ curv $\geq 1$.

$X = C_0 E$ Euclidean cone.
Alex. curv $X \geq 0$.

You can do hyperbolic cone too!
Say $X$ is Alexandrov, $\text{curv} X \geq 0$.
Consider a submetry (generalization of submersion) $\gamma : X \rightarrow Y$ whose $R$-balls map onto $R$-balls.

$\text{curv} Y \geq k$, $Y$ is also an Alexandrov space.

**Structure of Alexandrov spaces**
Surprisingly, not so bad! Infinitesimal $p \in X$, $T_p X$ tangent cone

\[
\lim_{\lambda \rightarrow \infty} \chi(x, p) = \frac{1}{\lambda} B(p, \lambda).
\]
You blow up about $p$, curvature $\rightarrow 0$, becomes scale invariant.

$T_p X = C \cdot S_p X$, $\text{Curv} S_p X \leq 1$; space of directions at $p$.

**What about local structure**

**Thm. (Perelman)** $\forall p \in X, \exists \varepsilon > 0$ s.t. $B(p, \varepsilon) \sim_{homo} T_p M$.

Proved by inverse induction on function.

Local structure $\Rightarrow$ deep understanding of topology.

**All these Alex space preserving transformations have many applications**

**Thm. (Stability Thm.) Perelman.**
Given $X, \text{curv} X \geq k$, $\exists \varepsilon = \varepsilon(X)$ s.t.

\[
d_{GH}(Y, X) < \varepsilon \quad \Rightarrow \quad \text{curv} Y \geq k.
\]

$X \sim Y$ homogeneous.
Manifolds with lower sectional curvature bounds

and

Alexandroff Geometry

Sec $M \geq k$

(M, g) complete Riem. n-manifold.

$S^2_k$ simply connected, constant curvature $k$

Equivalent (Toponogov)

\[ \begin{bmatrix} \frac{d}{dt} \end{bmatrix} \]

\[ \begin{bmatrix} \text{grad} \\ \text{dist} \end{bmatrix} \]

(classical $\triangle$ version)

\[ \begin{bmatrix} \text{good} \\ \text{dist} \end{bmatrix} \]

\[ \begin{bmatrix} \text{only} \\ \text{dist} \end{bmatrix} \]

(E) Embedding version (Balogh-Wu)
Critical Point for $p$

Regular

\[ \nabla \] regular direction

\[ \exists v \]

all min max from $q$ to $p$ make angle to $v > \frac{\pi}{2}$

$q$ critical if not regular:

(e.g. a max pt.)

\[ \exists v \]

\[ \forall v \]

Example:

\[ p \]

Note: $q$ regular $\Rightarrow \forall q' \in B(q, \varepsilon)$ regular

Convex combination

\[ \sum \alpha_i \text{ reg. directions is regular} \]

Isotopy:

\[ \text{KEY} \]

Can use any A cpt in place of $p'$

\[ \approx 6 \text{ minutes} \]
• Diam Sphere Thm
  \[ \text{Diam} \geq 1 \Rightarrow \text{diam} > \frac{\pi}{2} \]
  \[ M \cong S^n \text{ (topologically)} \]

• Betti # Thm
  \[ J C = C(n, k, D) : \text{sec}^2_k \text{ diam} > D \]
  \[ H_k(M) \text{ gen. by } \leq C \text{ elements} \]

  Related key observation: Weak Soul Thm
  \[ \text{sec} M > 0 \Rightarrow \text{noncpt } \Rightarrow \text{ Finite top type} \]
  \[ M \cong B(p, R) \cap M \]

  In fact \[ J R : \text{no ort pl, soul scale} \]
  \[ B(p, R) \]
  or else \[ \exists q_1, q_2, \ldots \]
  \[ 1p q_1 \perp 2p q_2 \] all p-prifnh

  \[ \text{all angles at } \]
  \[ p \geq \text{ some } \gamma \]

• Homotopy Finiteness
  \[ \exists C = C(n, k, D, v) : 0 \]
  \[ \text{at most } C \text{ homotopy types} \]

\[ \Delta(M) \]
\[ M \times M \]
\[ \approx 7 \text{ minutes} \]
\( Z \text{ opt metric} \)

\( A, B, C \subseteq Z \text{ closed} \)

\[ d_Z^2(A, B) = \inf_{x \in X} d_Z^2(x, A) + d_Z^2(x, B) \]

\( X, Y \text{ opt metric spaces} \)

\[ d^*_{ZH}(x, y) = \inf_{x \in X} d_Z^2(x, x') \]

\( Z = X \sqcup Y \)

Example

\[ d^*_{ZH}(\text{pt}, i'\text{pt}) = \sqrt{2} \]

\( X_n \rightarrow X \iff \exists \text{metric on } X \sqcup X_n = Z \text{ s.t. } d^*_{ZH}(x, x) \rightarrow 0 \)

Coarse:

\[ \sigma^*_{ZH} = 6 \text{ all opt metric space finite } \]

THM:

P: precompact \( G \rightarrow \)

\( \exists C(\infty) \text{ core function; i.e., uniform } \epsilon \text{-balls needed on all elements of } P \)

Example

\[ N^D_K \]

Examples
Preserved properties

- $X_i$ length $\Rightarrow \lim X_i = X$ length

- $\text{curv } X_i \geq k \Rightarrow \text{curv } X \geq k$, $X = \lim X_i$

- $\Rightarrow \text{ - p.}$

5. Alexandrov Spaces

(a) $X$ length space
(b) $\text{curv } X \geq k$

[ (c) $\dim X < \infty$ ]

Examples & Constructions

- $M$ Riem. m. $\Rightarrow \text{sec } M \geq k$
- $X = \lim M^n_i$, $\text{sec } M^n_i \geq k$
- $\Omega \subset \text{convex }, \partial \Omega \text{ curv } \geq 0$
- $\text{curv } E \geq 1 \Rightarrow C_0 E$ has curv $\geq 0$

$(C_1 E, C_0 E = S^1_E)$

- $\text{curv } E \geq 1 \Rightarrow \text{curv } E_1 \times E_2 \geq 1$ : $C_0 E_1 \times C_0 E_2 = C_0 (E_1 \times E_2)$

- $X \not\subseteq Y \Rightarrow X \not\subseteq Y$

- $X \cap \text{subdef } Y \not\subseteq Y \Rightarrow \text{curv } X \geq k$
- $\text{curv } Y \geq k$

Total 6 min page.
6 Structure

- **Infinitesimal**
  \[ x \in X \quad \frac{1}{2} B(x, \epsilon) \text{ unit ball in scaled space.} \]
  \[ T_x X = \lim_{\lambda \to \infty} (X_{\lambda x}) \text{ scale inv.} \]
  \[ \text{curvature } T_x X \geq 0 \]
  \[ \text{so } T_x X = \text{co } S_{\lambda x} \text{ unit sphere space of dir} \]
  \[ \text{curvature } S_{\lambda x} \geq 1 \]

- **Note**
  Any \( E \) with \( \text{curvature } \geq 1 \) (poincare)

- **Space and dir**
  \[ S_{\lambda x} = \hat{S}_{\lambda x} \text{ geodesic directions} \]

- **Local**
  \[ \exists \epsilon > 0 : B(x, \epsilon) \preceq T_x X (P) \text{ homeo.} \]
  \[ \text{Critical point theory?} \quad \text{Highly non-trivial} \quad \text{start with n-f.} \quad \text{induction} \]

- **Global structure**
  \[ \text{Stratification into manifolds} \]
  \[ \text{Metric or the ways} \]

- **Stability Thm**
  \[ (P) \forall \epsilon = \epsilon(X), \text{curv } X \preceq X, \text{dim } n \]
  \[ d_{H^1}(X, Y) < \epsilon \quad \text{curv } X \preceq X, \text{dim } Y = n \]
  \[ Y \preceq X \text{ homeo.} \]

- **Corollary**
  \[ H^D_{k,v}(n) \text{ contains at most finitely many homotopy types} \]
  \[ \text{Can } x \in \mathbb{N} \text{ dim } n? \]
Additional Applications

Note \( \text{rad} X \leq \text{diam} X \leq 2\text{rad} X \)

Vol \( D_k(n) \)

\[ \text{Vol} \quad V_4 \quad 1 \quad V \quad \min \quad \sec \]

Vol percolation
\[ \exists M \quad \text{vol} M^n \sim \text{vol} D^n \] (see M \( \leq K \) \( \text{Rad} M = \pi \))

such M are differ to \( S^n \), \( R^n \) \( (k \leq 4) \)

\& almost do to \[ 0 \]

GP \& PW \& diff, homeo, no seq eq.

Diff. Stability Problem
1.
2.

\[ \text{Diff} \& \text{Finikov} \text{ for } \prod_k X^n \text{ also } n = 4 \]

\( \text{Pro - Wilhelm} \)

(a) Diff eq limit for \( \prod_k X^n \)

(b) see M \( \geq 1 \), diam \( M > \frac{7}{4} \) diffe, \( S^n \)?

(\( \text{not collapse present} \))

\( \text{also Wilking} \)

9 pairs at } \frac{3}{2} \Rightarrow (n-2) \text{ such pair } \Rightarrow \text{diffe?} \)

9 minutes
COLLAPSE?

Nothing known? when $X$ has singularities?

Problem: $M^m \rightarrow X^l \neq \emptyset$ not regular.

1. Restrictions on $X$?
2. Almost Symmetry? Strata Fibration?
3. $X = \mathbb{P}^1$ $M^m$ almost non-curved

$S_0$: Nilpotent

Extended Bott-conjecture

Any $M \in S_0$ topologically elliptic
i.e. $F(M)\mathbb{Z}$ grows at most polynomially

Orbitsospores: when $\sec M \geq 0$

$G \times M \rightarrow M$

$M^l \rightarrow M/\Gamma$

Very nice structure

$(\text{Hilb-Gr})$ $M^l$-equiv. qft., see $M \geq 0$

$\downarrow \text{S^1-ISO}(H)$

$(M, \mathbb{S}^1)$ $\mathbb{S}^1$-equiv. diffeo to
lin action on $\mathbb{S}^3$, $\mathbb{C}P^2$

or sub $S^1 \mathbb{C}T^2$, where

$S^1 \mathbb{C}T^2 = T^2 \times S^3$

$S^3 \mathbb{C}T^2 = T^2 \times S^3$

$\text{Spin}(7)$, $\text{G}_2$

Orlicz

Raymond

Dinkhoff, 2006