Ricci Flow from Metrics with Isolated Conical Singularities

Felix Schulze

May 2, 2016

The presented results are joint work with P. Gianniotis. The Ricci flow on a manifold M is a smooth family of Riemannian metrics $(g_t)_{0\leq t\leq T}$ such that

$$\frac{d}{dt}g = -2\,\mathrm{Ric}(g).$$

Theorem 1 *Let (M, g_0) be a compact Riemannian manifold with isolated conical singularities at $\{z_i\}_{i=1}^Q \subset M$ modeled on metric cones $(\mathbb{S}^{n-1},\ g_i = dr^2 + r^2 g_{x_i})$ where $(\mathbb{S}^{n-1}, g_i)$ have curvature operator ≥ 1. Then there exists a smooth solution to the Ricci flow on M_1, $(g_t)_{0<t<T}$ such that*

1. *$|Rm(g(t))| \leq C/t$.*
2. *There exists $\psi : M \smallsetminus \{z_1,\dots,z_Q\} \to M$ diffeomorphism onto its image such that $\psi^* g(t) \to g_0$ smoothly away from $\{z_1,\dots,z_Q\}$.*
3. *$(M, d_{g(t)}) \to (M, d_{g_0})$ in Gromov-Hausdorff distance as $t \to 0$.*

Remark: Note that the assumption that the curvature operator of g_i is ≥ 1 can be weakened to $\geq 1-\varepsilon$.

Why are we interested in inital conical singularities?

Let $(N, (g_t)_{0\leq t\leq T})$ be a maximal solution Ricci flow with $T < \infty$. Assume that the flow develops a type I - singularity at (p_o, T). By a result by Müller-Enders-Topping the flow is modelled by a gradient of a shrinking soliton $(\overline{N}, \overline{g})$ near (p_0, T). Assume that $(\overline{N}, \overline{g})$ possesses a tangent cone C at infinity (Note that by a result by Munteau-Wang, if $Ric(\overline{g}) \to 0$ as $|p| \to \infty$, then $(\overline{N}, \overline{g})$ is smoothly asymptotic to a cone at ∞). Then

$$(\overline{N}, \overline{g}_t) \;\to\; C \text{ as } t \to 0.$$

It thus can be expected that (N, g_t) forms an isolated conical singularity at p_0, modelled on C, as $t \to T$.

Solutions Coming Out of Cones

- Bryant: Existence of rotationally symmetric expanding gradient solutions asymptotic to cones.
- Simon-Schulze: Let (M^n, g_0) with $curvop(g_0) \geq 0$ and $AVR > 0$: then there exists a Ricci flow $(g_t)_{t\geq 0}^n$ starting from g_0. Furthermore, any blowdown $(M, \lambda g\left(\frac{t}{\lambda}\right), p_0)$ converges subsequentially as $\lambda \to 0$ to an expanding gradient soliton coming out of the tangent cone at ∞ of (M, g_0).
- Deruelle: Existence of expanding solitons with positive curvature operator, asymptotic to cones. Uses a continuity continuity method and a uniqueness result for the symmetric case by Chodosh.
- Koch-Lamm: Let $(\mathbb{R}^n, g_0)$ such that $||g_0 - \delta||_{L^\infty} < \epsilon$. Then there exists a unique solution $(g_t)_{0\leq t<\infty}$ of Ricci-DeTurck flow
$$\frac{d}{dt}g = -2Ric(g) - \mathcal{L}_X g$$
with background δ such that $||g_t - \delta|| < C\varepsilon$ for all $t > 0$. Take $(\mathbb{S}^{n-1}, \tilde{g}_0)$ such that $||\tilde{g}_0 - g_{\mathrm{round}}||_{L^\infty} < \epsilon$. This implies that the conical metric $g_0 := (dr^2 + r^2 g)$ satisfies $||g_0 - \delta||_{L^\infty} < \varepsilon$. The uniqueness of the corresponding solution to Ricci-DeTurck flow and the invariance under parabolic rescalings implies that g_t is an expanding soliton coming out of g_0.

Proof idea of main result (compare Ilmanen-Neves-Schulze for network flow, and Begley-Moore in the case of Lagrangian mean curvature flow): Assume (M, g_0) has a conical singularity at z_0, modelled on $(C(\mathbb{S}^{n-1}), dr^2 + r^2 g))$ with $curvop(g) \geq 1$. By the result of Deruelle, there exists an expanding gradient soliton $(\hat{M}, \hat{g})$ coming out of the cone. Let $(\overline{M}, g_0^s)$ be (M, g_0) with $(\hat{M}, \hat{g})$ be glued into (M, g_0). We want to show that there exists a solution to Ricci flow, starting from g_0^s which exists for a time $T > 0$, independent of s. Note that away from z_0 we can use pseudolocality to get uniform estimates.

To control the flow in the region close to the initial conical singularity, we need a stability result for expanding gradient solitons. Recently Deruelle-Lamm extended the L^∞-stability of Koch-Lamm to expanders with positive curvature operator.

Aim: Localise Deruelle-Lamm to show that solutions stay close to the expanding soliton around the initial singularity. Control the 'boundary values' via pseudolocality. Problem: the stability result is for Ricci-DeTurck flow, so one also needs to control the diffeomorphisms relating Ricci flow to Ricci-DeTurck flow.

We also show:

Theorem 2 *Let (M, g_t) be the solution constructed in the first theorem starting at (M, g_0). Then any sequence of forward rescalings*

$$(M, \lambda_i g(t/\lambda_i), z_l) \to (\hat{M}_l, \hat{g}_t^l, 0)$$

as $\lambda_i \to \infty$, where $(\hat{M}_l, \hat{g}_t^l)$ is the expanding solution glued into g_0 around z_l.