MSRI
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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Justin Hilburn **Email/Phone:** jhilburn@uoregon.edu

Speaker's Name: Hiraku Nakajima

Talk Title: Coulomb Branches of 3d N=4 Supersymmetric quantum field theories

Date: 11 / 18 / 2014 **Time:** 11 : 00 am pm (circle one)

List 6-12 key words for the talk: Supersymmetry, Higgs Branch, Coulomb Branch, Hyperkahler, Monopole Formula, Quantum Field Theory

Please summarize the lecture in 5 or fewer sentences:
From a 3d N=4 supersymmetric gauge theory physicists can construct two hyperkahler manifolds: the Higgs and Coulomb branches. Unfortunately until this time only the Higgs branch has a mathematically satisfactory definition. Nakajima gave a conjectural description of the coordinate ring of the coulomb branch in terms of the cohomology of a moduli space of monopoles.

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
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☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Nakajima — Coulomb Branches of 3d $N=4$ Gauge theories and Donaldson–Thomas invariants

(Braverman – Finkelberg)

(G reductive group $/ \mathbb{C}$
M symplectic representation of G ← often $N \oplus N^*$ where N is a rep $/ \mathbb{C}$.

$\Longrightarrow$

3D $N=4$ SUSY gauge theory

$\Longrightarrow$

2 Hyperkähler manifolds
- M_H Higgs branch
- M_C Coulomb branch

M_H

$$M_H := M /\!/\!/ G = \mu^{-1}(0) /\!/ G = \operatorname{Spec}(\mathbb{C}[\mu^{-1}(0)]^G)$$

↑ symplectic quotient

$$\mu^{-1}(0)^{\text{free}} / G \underset{\text{open}}{\subset} M /\!/\!/ G$$

↑ holomorphic symplectic manifold

Properties

① $\mathbb{C}^* \curvearrowright M /\!/\!/ G$ symplectic form wt > 0

↑ induced by scalar multiplication on M

② If G has nontrivial character $\chi : G \to \mathbb{C}^\times$

Can consider $M /\!/\!/_\chi G = \mu^{-1}(0) /\!/_\chi G = \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(0)]^{n\chi}\right)$

↑ G.I.T quotient

$M///_\alpha G \longrightarrow M///G$ is often a symplectic resolution.

(3) Suppose M is a representation of a larger group $\tilde{G} \triangleright G$. Then $G_F = \tilde{G}/G$ acts on $M///_0 G$.
$\uparrow$ Flavour symmetry

(4) $M = N \oplus N^*$ quantization of $M///G$ by quantum Hamiltonian of $D(N)$ by G
$\uparrow$ differential operators on N

Goal: Give a definition of Coulomb branch.
Note $\dim_{\mathbb{C}} \mathcal{M}_C = 2 \cdot \operatorname{rank} G$.
Issue: quantum corrections of metric on $\mathcal{M}_C$.

Remark Symplectic duality : (Braden – Proudfoot – Licata – Webster)

$$\begin{array}{ccc} \text{quantization}(\mathcal{M}_H)\text{-mod} & & \text{quantization}(\mathcal{M}_C)\text{-mod} \\ \cup & \text{Koszul dual} & \cup \\ \mathcal{O}_H & \longleftrightarrow & \mathcal{O}_C \end{array}$$

For Type A flag variety this recovers classical Koszul duality on $\mathcal{O}_{\mathfrak{sl}_n}$

Properties (expected):

① M_C affine algebraic variety $\circlearrowleft \mathbb{C}^\times$
(symplectic on open part)

Examples

a) Abelian gauge theory

$$1 \to G = T^d \hookrightarrow T^n \twoheadrightarrow T_F^{n-d} \to 1$$

G: gauge; $T^n = (\mathbb{C}^\times)^n$; T_F^{n-d}: flavor

$$M_H = M /\!/\!/ T^d \circlearrowleft T_F^{n-d}$$

hypertoric manifold (Bielawski – Dancer)

Have dual tori

$$1 \to (T_F^{n-d})^\vee \to (T^n)^\vee \to (T^d)^\vee \to 1$$

canonically self dual

$$M_C = M /\!/\!/ (T_F^{n-d})^\vee$$

b) $G = GL(k)$, $M = 0$

understand from classification of hyper kähler manifold with large sym

$M_H = 0$

$M_C = \begin{cases} \text{Seiberg-Witten for } k=2 \\ \text{Chalmers-Hanany for } k \geq 2 \end{cases}$

D-branes

$=$ moduli space of charge k BPS $SU(2)$ monopole

use Nahm's transform

Donaldson

$\hookrightarrow = \left\{ \begin{array}{l} \text{based maps } \mathbb{P}^1 \to \mathbb{P}^1 \\ \deg k \end{array} \right\}$

Bielawski

$=$ symplectic reduction $T^*GL(k)$
by $N_- \times N_-$ at (y, y)

where (x,y,h) is an sl_2 triple
for regular nilpotent x.

c) quiver gauge theory (finite type)
Γ = ADE dynkin diagram
$Q = (Q_0, Q_1)$ quiver

$V = \bigoplus_{i \in Q_0} V_i$ $\qquad W = \bigoplus_{i \in Q_0} W_i$.

dim W dominant integral wt

$G = \prod_i GL(V_i)$

$G_H = \prod_i GL(W_i)$

μ = dim V − C dim V ← weight

$M = N \oplus N^*$

$N = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{s(h)}, V_{t(h)}) \oplus \bigoplus_i \mathrm{Hom}(W_i, V_i)$

$M_H = M_0(V, W)$
quiver variety of finite type

expected answer:

$= M_C \subset Gr_H$ $\qquad$ affine grassmannian
H = (adjoint) group of type Γ

$\overline{H(\theta) t^\lambda}$
$\cap$
transverse slice to
$H(0) t^\mu$

② $1 \to G \to \hat{G} \to G_F \to 1$

For λ_F coweight of G_F

$\rightsquigarrow \exists\, \mathcal{M}_C^{\lambda_F} \to \mathcal{M}_C \qquad \mathcal{L}_{\lambda_F}$ line bundle

$\to$

In abelian case these are the G.I.T. quotients $\mathcal{M}/\!/_{\lambda_F} T_F^\vee$

③ $\mathrm{Hom}(\pi_1(G), \mathbb{C}^*) \curvearrowright \mathcal{M}_C$

can be extended to non-abelian group action

	$\mathcal{M}_H$	$\mathcal{M}_C$
G_F	action	line bundle
G	line bundle	action

Proposal of the definition of the Coulomb branch $\mathcal{M}_C$

Need $\mathcal{M} = N \oplus N^*$

Gr = affine grassmannian of G

$= G((z)) / G[[z]]$

$$G[[z]] \backslash \mathrm{Gr} = \left\{ \begin{array}{l} G\text{-bundles on } D = \mathrm{Spec}\, \mathbb{C}[[z]] \\ + \text{ trivialization on } \hat{D} = \mathrm{Spec}\, \mathbb{C}((z)) \end{array} \right\}$$

$$R = \left\{ (P_1, P_2, \varphi, s_1, s_2) \;\middle|\; \begin{array}{l} s_i \in H^0(P_i \times_G N) \\ \varphi(s_1) = s_2 \end{array} \right\}$$

convolution diagram

$$\begin{tikzcd} & \{ (P_1, P_2, P_3, \varphi_{12}, \varphi_{23}, s_{12}, s_{23}) \mid \varphi_{12}(s_{12}) = s_{23} \} \arrow[dl, "p_{12}"'] \arrow[d, "p_{23}"] \arrow[dr, "p_{13}"] & \\ R & R & R \end{tikzcd}$$

$$(p_{13})_* (p_{12}^* \circ p_{23}^*)$$

$$\rightsquigarrow \quad H_*^{BM}(R) \otimes H_*^{BM}(R) \to H_*^{BM}(R)$$

<u>Proposal</u> $\quad \mathbb{C}^* \curvearrowright \mathbb{C}[M_C] = H_*^{BM}(R)$

equivariant parameter = deformation parameter

☆ Natural quantization

$$\mathbb{C}^* \curvearrowright R \quad \mathbb{C}^*\text{-equiv homology} \quad \hbar = H^{BM}_{\mathbb{C}^*,*}(R)$$

☆ $M = 0 \qquad G = GL(k) \qquad$ ex b)

$H_*(Gr)$ — Bezrukavnikov, Finkelberg, Mirkovic

Stratification of Gr

$\rightsquigarrow$ " of R

$\rightsquigarrow$ LES for $H_*^{BM}(R)$

get formula for dimensions called monopole formula Hanany