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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Justin Hilburn **Email/Phone:** jhilburn@uoregon.edu

Speaker's Name: Xinwen Zhu

Talk Title: Commutativity constraints revisited

Date: 11 / 21 / 14 **Time:** 2 : 00 am / [✓ pm] (circle one)

List 6-12 key words for the talk: Geometric Satake, Tannakian formalism, Commutativity, Affine Grassmannian, p-adic field, Perverse Sheaf

Please summarize the lecture in 5 or fewer sentences: The standard tool for proving commutativity of the convolution product in geometric Satake is the fusion product on the Beilinson-Drinfeld Grassmannian. Unfortunately this is not available for p-adic groups. In this talk Zhu presented a new more explicit proof of commutativity that does generalize to p-adic groups.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

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Xinwen Zhu - Commutativity constraints revisited

G/k $\quad k = \bar{k}$ $\quad \mathcal{O} = k[[t]]$ $\quad F = k((t))$ $\quad D = \operatorname{Spec} \mathcal{O}$ $\quad D^\times = \operatorname{Spec} F$

$$\rightsquigarrow \quad Gr_G = \left\{ (E, \beta) \;\middle|\; \begin{array}{l} E \quad G\text{-bundle on } D \\ \beta: E|_{D^\times} \cong E_0|_{D^\times} \end{array} \right\}.$$

$$= G(F)/G(\mathcal{O}) = \varinjlim Gr_\mu \circlearrowleft G(\mathcal{O})$$

$$Sat_G = (P_{G(\mathcal{O})}(Gr), *)$$

convolution

Geometric Satake iso:

Sat_G is a Tannakian cat $\left\{ \begin{array}{l} \text{tensor cat} \\ H^*: Sat_G \to Vect \\ \text{fiber functor} \end{array} \right.$

$$H: Sat_G \cong Rep(\check{G})$$

The proof of this relies on B-D Grassmannian

$$\begin{array}{c} Gr_{X^n} \\ \downarrow \\ X^n \end{array} = \left\{ (E, \beta) \;\middle|\; \begin{array}{l} E \quad G\text{-bundle on } X \\ \beta: E|_{X - \{x_1, \ldots, x_n\}} \cong E_0|_{X - \{x_1, \ldots, x_n\}} \end{array} \right\}$$

$$Gr_{X^2}|_{(x,y)} = \left\{ \begin{array}{ll} Gr \times Gr & x \neq y \\ Gr & x = y \end{array} \right.$$

$\leftarrow$ only OK because of infinite dimensionality!!

$\mathcal{F} \in \mathrm{Sat}_G \rightsquigarrow \mathcal{F}_X$ on Gr_X

$$j_{!*}\left(\mathcal{F}_X \boxtimes \mathcal{F}'_X \big|_{X^2-\Delta}\right)\Big|_{\Delta} = (\mathcal{F} * \mathcal{F}')_X$$

Get commutativity

$$\mathcal{F} * \mathcal{F}' \cong \mathcal{F}' * \mathcal{F} \qquad (1)$$

and

$$H(\mathcal{F} * \mathcal{F}') \simeq H(\mathcal{F}) \otimes H(\mathcal{F}') \qquad (2)$$

Purpose of this talk - Give new proofs of (1) and (2) without BD-Grassmannian.

$k = \mathbb{F}_q$ $\qquad F = \begin{cases} \mathbb{F}_q((t)) \\ \mathbb{Q}_p \end{cases}$

$$C(G(\mathcal{O}) \backslash G(F) / G(\mathcal{O})) \simeq K(\mathrm{Rep}\, \check{G})$$

<u>Thm (Z)</u> Same statement holds for G split reductive group.

$$\mathrm{Gr}^W(R) = \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ G-bundle on } W(R) \\ \mathcal{E}|_{W(R)[\frac{1}{p}]} \cong G|_{W(R)[\frac{1}{p}]} \end{array} \right\}$$

$\nearrow$ perfect ring

<u>Thm (Z)</u> Gr^W is represented as an Ind-perfect alg space

$$\mathrm{Gr}^W = \varinjlim \mathrm{Gr}_\lambda$$

Each Gr_λ is perfection of a proper alg space.

$H_{G(O)} : \mathrm{Sat}_G \longrightarrow R_G \times R_G$ bimodules $R_G = H^*_{G(O)}(pt)$

$H_{G(O)}(\mathcal{F} * \mathcal{F}') = H_{G(O)} \otimes_{R_G} H_{G(O)}(\mathcal{F})$

Lemma The bimodule structures are the same $C\otimes_{R_G}$ - monoidal structure of H^*.

Ex $E \hookrightarrow E'$ v.b. on $R[[t]]$ E'/E line bundle on $R[[t]]/t$

$c(E/t) = c(E'/t)$ in $H(\operatorname{Spec} R)$.

$\theta : G \to G$ Cartan involution

$\theta(\lambda) = -w_0(\lambda)$

$\theta : G(F) \longrightarrow G(F)$

$g \mapsto \theta(g)^{-1}$

$G(O) \times^{u} G(O) \to G(O) \times^{u} G(O)$

$\theta^* : \mathrm{Sat}_G \to \mathrm{Sat}_G$ anti-monoidal

$\theta^*(\mathcal{F} * \mathcal{F}') \cong \theta^*(\mathcal{F}') * \theta^*(\mathcal{F})$

Informally

$\theta : G(O)\backslash G(F)/G(O) \to G(O)\backslash G(F)/G(O)$

Formally $Gr^{op} = G(O)\backslash G(F) \swarrow \mathrm{Sat}^{op}_{O}$ $G(O)\backslash G(F) \xrightarrow{\theta} G(F)/G(O)$

$$\theta^* : P_{G(O)}(Gr) \underset{\text{antimonoidal}}{\overset{\theta}{\simeq}} P_{G(O)}(Gr^{op}) \underset{\text{monoidal}}{\overset{\text{Taut}}{\simeq}} P_{G(O)}(Gr)$$

Enough to construct

$e : \theta \simeq \mathrm{Id}$ $\qquad e_\mu : \theta^* IC_\mu \simeq IC_\mu$

$$\theta^* IC_\mu \big|_{Gr^0_\mu} \simeq \mathbb{C}[\langle 2\rho,\mu\rangle] \xrightarrow[(-1)^{\langle 2\rho,\mu\rangle}]{} \mathbb{C}[\langle 2\rho,\mu\rangle] = IC_\mu \big|_{Gr^0_\mu}$$

The construct is

$$\begin{tikzcd} \mathcal{F} * \mathcal{G} \arrow[r, "\overset{e}{\simeq}"] & \theta^*(\mathcal{F} * \mathcal{G}) \simeq \theta^*(\mathcal{G}) * \theta^*\mathcal{F} \\ & \overset{e}{\simeq} \mathcal{G} * \mathcal{F} \end{tikzcd}$$

+ Koszul sign change.

Ex $G = GL_2$

$$\begin{tikzcd} IC_{\omega_1} * I_{\omega_1} \arrow[r, equal] \arrow[d, "\wr"] & IC_{\omega_1} * IC_{\omega_1} \arrow[d, "\wr"] \\ IC_{2\omega_1} \oplus IC_{\omega_2} \arrow[r, "\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}"] & IC_{2\omega_1} \oplus IC_{\omega_2} \end{tikzcd}$$

Need

$$\begin{tikzcd} H^*(\mathcal{F} * \mathcal{F}') \arrow[r, "\simeq"] \arrow[d, "\wr\, M"] & H(\mathcal{F}' * \mathcal{F}) \arrow[d, "\wr\, M"] \\ H^*(\mathcal{F}) \otimes H^*(\mathcal{F}') \arrow[r] & H^*(\mathcal{F}') \otimes H^*(\mathcal{F}) \end{tikzcd} \quad (1)$$

Let $\theta : H^*(\mathcal{F}) \overset{\gamma}{\simeq} H^*(\theta^*\mathcal{F}) \simeq H^*(\mathcal{F})$

(2) θ acts as $(-1)^i$ on $H^i(\mathcal{F})$

(2) ⇒ (1) $IH(Gr_\mu) = H(IC_\mu[\langle -2\rho,\mu\rangle])$

$\theta^*: IH(Gr_\mu) \longrightarrow IH(Gr_\mu)$

(2)' θ^* acts as $(-1)^j$ on $IH^{2j}(Gr_\mu)$

(Lusztig - Yau)

$$\begin{tikzcd} Gr_\mu \supset Gr^\circ_\mu = G(\mathcal{O})\, t^\mu\, G(\mathcal{O})/G(\mathcal{O}) \arrow[d] \\ G/P_\mu \end{tikzcd}$$

claim $\theta^* = (-1)^j : H^{2j}(Gr^\circ_\mu) \to H^{2j}(Gr^\circ_\mu)$

$$\begin{tikzcd} Gr_\mu & Gr^{\circ\prime}_\mu \arrow[r, "\theta"] \arrow[d] & Gr^\circ_\mu \arrow[d] \\ & G/P_{w_0\mu} & G/P_\mu \end{tikzcd}$$

$\mathbb{C}_\mu - IC_\mu[(2\rho,\mu)]$

$e' : \theta^* \mathbb{C}_\mu \simeq \mathbb{C}_\mu \qquad \theta^* \mathbb{C}_\mu|_{Gr^\circ_\mu} = \mathbb{C} = \mathbb{C}_\mu|_{Gr_\mu}$

$e'|_\lambda \quad \theta^* \mathbb{C}_\mu|_\lambda \simeq \mathbb{C}_\mu|_\lambda$

(2) ⟺ (2') $\qquad H^{2j}(e'|_\lambda) = (-1)^j$

Define $\quad P_{\lambda,\mu}(q) = \sum \dim H^{2i}(\mathbb{C}_\mu|_\lambda)\, q^j$

$P^{\theta\sigma}_{\lambda,\mu}(q) = \sum tr(e'|_\lambda : H^{2i}(\mathbb{C}_\mu|_\lambda))\, q^j$

$$(2'') \Leftrightarrow (2''') \qquad \underset{\substack{\uparrow \\ \text{Lusztig} \\ \text{Vogan}}}{P^{\Delta,\sigma}_{\lambda,\mu}} \underset{\substack{\uparrow \\ \text{Proved by} \\ \text{Lusztig - Yun}}}{=} \underset{\substack{\uparrow \\ \text{Kazhdan} \\ \text{Lusztig}}}{P_{\lambda,\mu}}(-q)$$