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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: KAROL KOZIOL Email/Phone: (315)569-6968

Speaker's Name: ANA CARAIANI

Talk Title: PATCHING AND p-ADIC LOCAL LANGLANDS

Date: 8 / 14 / 14 Time: 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: p-ADIC LOCAL LANGLANDS CORRESPONDENCE, BREUIL-SCHNEIDER CONJECTURE,

Please summarize the lecture in 5 or fewer sentences: IN THIS TALK, AN OUTLINE IS GIVEN OF A POTENTIAL CANDIDATE FOR THE p-ADIC LOCAL LANGLANDS CORRESPONDENCE FOR $GL_n(F)$, F A FINITE EXTENSION OF $\mathbb{Q}_p$. AS AN APPLICATION, ~~MANY~~ NEW CASES OF THE BREUIL-SCHNEIDER CONJECTURE ARE PROVED

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

PATCHING + p-ADIC LOCAL LANGLANDS

– A. CARAIANI

JOINT W/ EMERTON, GEE, GERAGHTY, PASKUNAS, SHIN

– APPROACH TO p-ADIC LOCAL LANGLANDS USING GLOBAL METHODS

INTRO

LET $F/\mathbb{Q}_p$ BE A FINITE EXT.

CLASSICAL LLC

$$\left\{ \begin{array}{c} \text{IRRED SMOOTH} \\ \text{REPS OF} \\ GL_n(F) \end{array} \right\}/\sim \quad \overset{\text{BIJ}}{\longleftrightarrow} \quad \left\{ \begin{array}{c} n \text{ DIM'L, FROB-SS} \\ \text{WD REPS OF} \\ W_F \end{array} \right\}/\sim$$

$$\pi \longmapsto \rho(\pi)$$

$$\pi(\rho) \longmapsfrom \rho$$

NB "SMOOTH" ON LHS + "WD REP" ON RHS REMOVES TOPOLOGY OF COEFF FIELD, SO WE CAN TAKE ANY CHAR 0 FIELD (EG, $\mathbb{C}$, $\overline{\mathbb{Q}}_\ell$)

ASSUME COEFF FIELD IS $\overline{\mathbb{Q}}_\ell$

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• $\ell \neq p$: GIVEN

$\rho : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ CTS, CAN ASSOCIATE A WD REP DENOTED $WD(\rho)$ BY ℓ-ADIC MONODROMY

TO $WD(\rho)$ CAN ASSOCIATE $\pi(\rho)$

IN THIS CASE, LLC IS A THM DUE TO HARRIS-TAYLOR AND HENNIART

• $\ell = p$: LET

$\rho : G_F \longrightarrow GL_n(E)$, $E/\mathbb{Q}_p$ FINITE

BE CTS. IN GENERAL, NO WAY TO ASSOCIATE TO ρ A WD REP. BUT IF ρ IS DE RHAM, WE CAN ASSOCIATE ~~TO~~ $WD(\rho)$, AND THEN $\pi(\rho) \otimes \pi_{ALG}(\rho)$, BUT

$$\underset{\text{SMOOTH}}{\pi(\rho)} \otimes \underset{\text{ALGEBRAIC}}{\pi_{ALG}(\rho)}$$

$$\rho \rightsquigarrow WD(\rho)$$

FORGETS THE p-ADIC HODGE FILTRATION.

BREUIL: THERE SHOULD BE A $GL_n(F)$ - INVT NORM ON $\pi(\rho) \otimes \pi_{ALG}(\rho)$ WHICH REMEMBERS HODGE FILTRATION

$\to$ CAN THEN COMPLETE $\pi(\rho)$ TO $\hat{\pi}(\rho)$, A BANACH SPACE REP

FOR $F = \mathbb{Q}_p$, $n = 2$ WE HAVE

THM (BERGER, BREUIL, COLMEZ, DOSPINESCU, EMERTON, KISIN, PASKUNAS)
THERE IS A BIJ

$$\left\{ \rho : G_{\mathbb{Q}_p} \xrightarrow[\text{CTS}]{} GL_2(E) \right\} / \sim \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{CERTAIN} \\ \text{ADMISSIBLE UNITARY} \\ p\text{-ADIC BANACH SPACE} \\ \text{REPS OF } GL_2(\mathbb{Q}_p) \text{ OVER } E \end{array} \right\} / \sim$$

$$\rho \longmapsto \hat{\pi}(\rho)$$

RMKS 1) COMPATIBLE W/ RED mod p + DEFORMATIONS

2) COMPATIBLE W/ CLASSICAL LLC

$$(\hat{\pi}(\rho))_{L\text{-}ALG} \neq 0 \iff \rho \text{ IS DE RHAM W/ DISTINCT HT WTS}$$

IN THIS CASE $(\hat{\pi}(\rho))_{L\text{-}ALG} \cong \pi(\rho) \otimes \pi_{ALG}(\rho)$

($v \in \Pi$ IS LOCALLY ALG IF $\exists\, K \subset G$ OPEN ST $K \to \Pi : k \mapsto kv$ IS ALG.)

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3) COMPATIBLE W/ GLOBAL PICTURE : COMPLETED COH.

Y = MODULAR CURVE , $K^p \subset GL_2(\mathbb{A}_f^p)$ COMPACT OPEN ("TAME LEVEL")

$K_p \subset GL_2(\mathbb{Q}_p)$ COMPACT OPEN

$$\hat{H}^1(K^p) := \Big(\varinjlim_{K_p} H^1_{\acute{e}t}(Y(K^pK_p)\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathcal{O}_E)\Big)^{\wedge}\big[1/p\big]$$

p-ADIC COMPLETION

$G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)$

$\Rightarrow$ $\hat{H}^1 := \varinjlim_{K^p} \hat{H}^1(K^p)$ HAS AN ACTION OF $GL_2(\mathbb{A}_f^p)\times GL_2(\mathbb{Q}_p) \times G_{\mathbb{Q}}$

LET $r: G_{\mathbb{Q}} \longrightarrow GL_2(E)$ BE A GLOBAL REP ST r IS CTS, UNRAMIFIED ALMOST EVERYWHERE, + OTHER TECHNICAL ASSUMPTIONS. THEN

<u>THM</u> (EMERTON)

$$\mathrm{HOM}_{G_{\mathbb{Q}}}(r, \hat{H}^1) \cong \hat{\pi}(r|_{G_{\mathbb{Q}_p}}) \otimes \bigotimes_{\ell\neq p}' \pi(r|_{G_{\mathbb{Q}_\ell}})$$

$\Rightarrow$ p-ADIC LLC CAN BE USED TO PROVE CASES OF FONTAINE-MAZUR CONJ ("MODULARITY THM") :

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IF $r|_{G_{\mathbb{Q}_p}}$ IS DE RHAM W/ DISTINCT HT WTS,

THEN $\left(\hat{\pi}(r|_{G_{\mathbb{Q}_p}})\right)^*_{L\text{-ALG}} \neq 0$

$$\Longleftrightarrow \quad \mathrm{HOM}_{G_{\mathbb{Q}}}(r, (\hat{H}^1)_{L\text{-ALG}}) \neq 0$$

CLASSICAL MOD FORMS OF WT ≥ 2 $\otimes$ ATTACHED GALOIS REP

(CLASSICAL MOD FORMS CONTRIBUTE TO H^1_{ET}, BUT MAYBE W/ COEFFS IN A LOCAL SYSTEM $\longrightarrow$ ALL INCLUDED IN $\hat{H}^1$)

FOR $F \neq \mathbb{Q}_p$, GENERAL n, MOST TECHNIQUES FOR

$$\rho \longleftrightarrow \hat{\pi}(\rho)$$

BREAK DOWN.

STILL HAVE GLOBAL AUT FORMS + CLASSICAL LLC, SO IF

$$\rho : G_F \longrightarrow GL_n(E)$$

IS CTS, DE RHAM W/ DISTINCT HT WTS, CAN FORM

$$BS(\rho) := \pi(\rho) \otimes \pi_{ALG}(\rho)$$

<u>CONJ</u> (BREUIL - SCHNEIDER) LET π BE A LOCALLY ALG REP OF $GL_n(F)$, IRRED. THEN

π ADMITS A UNITARY COMPLETION $\Longleftrightarrow$ $\pi = BS(\rho)$ FOR SOME ρ AS ABOVE

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"$\Rightarrow$" KNOWN BY WORK OF HU

"$\Leftarrow$" KNOWN FOR SUPERCUSPIDALS, STEINBERG (SORENSEN)

$\boxed{\text{THM IF } \rho \text{ IS GENERIC, POT CRYSTALLINE, AND LIES ON AN AUTOMORPHIC COMPONENT OF A LOCAL DEFORMATION RING, THEN } BS(\rho) \text{ ADMITS AN ADMISSIBLE UNITARY COMPLETION}}$

EX IF $F/\mathbb{Q}_p$ UNRAM, ρ CRYS, HT WTS IN EXTENDED FL RANGE, GET UNRAMIFIED PRINCIPAL SERIES.

IDEA LET $\bar{\rho} : G_F \longrightarrow GL_n(\mathbb{F})$ BE CTS

$\rightsquigarrow$ GET $R_{\bar{\rho}}$ (FRAMED) UNRESTRICTED LOCAL DEF RING

1) USE TAYLOR-WILES PATCHING FOR COMPLETED COH.

GET A MODULE M_∞ OVER $R_\infty := R_{\bar{\rho}}[[z_1, \dots, z_r]]$

W/ AN ACTION OF $GL_n(F)$ ST FIBERS OVER CLOSED PTS OF $R_\infty[1/p]$ ARE UNITARY ADM p-ADIC BANACH SPACES

2) LET σ BE A LOC. ALG. REP OF $GL_n(\mathcal{O}_F)$ ("TYPE")

USE MOD LIFTING THMS TO SHOW THERE ARE NONZERO LOC ALG VECTORS

$$M_\infty(\sigma) := \mathrm{Hom}^{cts}_{GL_n(\mathcal{O}_F)}(M_\infty, \sigma^\vee)^\vee$$

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3) INERTIAL LLC

$\forall \tau : I_F \longrightarrow GL_n(E)$ INERTIAL TYPE

$\exists\ \sigma(\tau)$ SMOOTH IRREP OF $GL_n(\mathcal{O}_F)$ ST

$\mathrm{HOM}_{GL_n(\mathcal{O}_F)}(\sigma(\tau), \pi) \neq 0 \iff \pi = \pi(\rho)$ WITH $\rho|_{I_F} \cong \tau$

4) INTERPOLATE LL

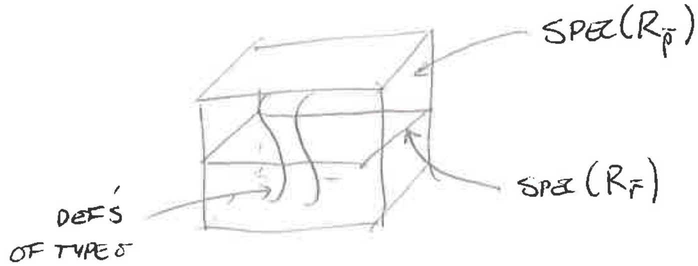
IDEAS FOR 1) GLOBALIZE: $\tilde{F}, \tilde{\wp}$ GLOBAL,

$\bar{r} : G_{\tilde{F}} \longrightarrow GL_n(\mathbb{F})$

SUCH THAT

- $F \cong \tilde{F}_{\tilde{\wp}}$
- $\bar{r}|_{G_F} \cong \rho$
- $\bar{r}$ AUTOMORPHIC

GET $R_{\bar{\rho}} \longrightarrow R_{\bar{r}}$



$\tilde{G}/\tilde{F}$ DEF UNITARY GRP $\rightsquigarrow$ ALG AUT FORMS

CLASSICAL: PATCH ALG MODULAR FORMS OF TYPE σ W/ VARYING TAME LEVEL

$\rightsquigarrow M_\infty(\sigma)$ MODULE OVER $R_\infty(\sigma) \cong R_{\bar{\rho}}(\sigma)[\![\ldots]\!]$

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OUR SETTING:

PATCH COMPLETED COH

$$M_N = S(U(Q_N, K_N), \mathcal{O}_E/\varpi^N)^\vee \quad \text{LEVEL } N$$

TAME LEVEL, VARIES

LEVEL AT p

+ KEEP TRACK OF PARTIAL $GL_n(F)$-ACTION

THIS GIVES M_∞, AND WE SET $V(\rho) = M_\infty[1/p] \otimes_{R_\infty, \rho} E$

$$\rightsquigarrow \quad \mathrm{HOM}_K(\sigma, V(\rho)) = \mathrm{HOM}_G(c\text{-}\mathrm{IND}_K^G(\sigma), V(\rho)) = \mathrm{HOM}(BS(\rho), V(\rho))$$

$\neq 0$