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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: KAROL KOZIOL Email/Phone: (315)569-6968

Speaker's Name: SARAH ZERBES

Talk Title: EULER SYSTEMS AND THE BIRCH-SWINNERTON-DYER CONJECTURE

Date: 8/15/14 Time: 9:30 am / pm (circle one)

List 6-12 key words for the talk: EULER SYSTEMS, ELLIPTIC CURVES, BSD CONJECTURE

Please summarize the lecture in 5 or fewer sentences: IN THIS TALK, A VARIANT OF THE BIRCH-SWINNERTON-DYER CONJECTURE IS STATED. USING EULER SYSTEMS AND p-ADIC DEFORMATIONS, SEVERAL CASES OF THIS VARIANT ARE PROVEN

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

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8/15 EULER SYSTEMS AND THE BSD CONJECTURE

– S. ZERBES

JOINT W/ KINGS, LEI, LOEFFLER

I BSD CONJECTURE

E ELLIPTIC CURVE $/\mathbb{Q}$. THEN $E(\mathbb{Q}) \cong \Delta \times \mathbb{Z}^{r_E}$ ← RK, $|\Delta| < \infty$

ASSOCIATE TO E THE L-FUNCTION

$L(E,s)$ = INFINITE PRODUCT OF LOCAL TERMS, CONVERGES FOR $\mathrm{Re}(s) > 3/2$

$\boxed{\text{THM (WILES, BCDT)} \quad L(E,s) \text{ HAS ANALYTIC CTATION TO ALL } s \in \mathbb{C}}$

BSD CONJ: · $\mathrm{ORD}_{s=1} L(E,s) = r_E$

· LEADING TERM AT $s=1$ HAS AN EXPRESSION IN TERMS OF ARITHMETIC INVT'S OF E (INCLUDING $|\text{Ш}(E/\mathbb{Q})|$)

GENERALIZATIONS: ρ = ARTIN CHAR OF $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ FACTORING THROUGH F

$\leadsto$ CAN CONSTRUCT $L(E,\rho,s)$

IF ρ IS 1-DIML OR ODD + 2-DIML, $L(E,\rho,s)$ HAS ANALYTIC CTATION TO ALL $s \in \mathbb{C}$

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BSD_p - CONJ : • $\mathrm{ORD}_{s=1} L(E, \rho, s) = \mathrm{RK}\,(E(F)[\rho])$

p - ISOTYPIC

THM (KOLYVAGIN - KATO)
IF $L(E,1) \neq 0$, THEN $E(\mathbb{Q})$ IS FINITE, AND $Ш(E/\mathbb{Q})[p^\infty]$ IS FINITE FOR ALMOST ALL p

WANT TO GENERALIZE THIS TO BSD_ρ CASE

KEY IDEAS (FOLLOWING KATO)

1) MAKE PROBLEM p-ADIC

↳ ATTACH TO E A p-ADIC REP OF $G_{\mathbb{Q}}$:

$$T_p E = \varprojlim E(\overline{\mathbb{Q}})[p^n] \qquad (\cong \mathbb{Z}_p^2)$$

$$V_p E = T_p E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

CONSIDER $G_{\mathbb{Q}}$ - COHOMOLOGY

$$\begin{array}{c} H^1(\mathbb{Q}, V_pE) := H^1(G_{\mathbb{Q}}, V_pE) \\ \cup \\ \mathrm{Sel}(\mathbb{Q}, V_pE) \end{array}$$

FACTS
- $E(\mathbb{Q}) \otimes \mathbb{Q}_p \hookrightarrow \mathrm{Sel}(\mathbb{Q}, V_pE)$
- QUOTIENT RELATED TO $Ш(E/\mathbb{Q})[p^\infty]$

$\Longrightarrow$ SUFFICES TO SHOW "$L(E,1) \neq 0 \Longrightarrow \mathrm{Sel}(\mathbb{Q}, V_pE) = 0$"

2) USE EULER SYSTEM (ES) FOR $V_p E$

= COLLECTION OF CLASSES $(z_m)_{m \geq 1}$, $z_m \in H^1(\mathbb{Q}(\mu_m), V_p E)$

SATISFYING CERTAIN COMPATIBILITIES UNDER CORESTRICTION MAPS (FOR $\mathbb{Q}(\mu_m) \hookrightarrow \mathbb{Q}(\mu_{m'})$)

ES IS RELATED TO $L(E,1)$

$\exists$ LINEAR FUNCTIONAL (BLOCH - KATO EXP.)

$$\exp^* : H^1(\mathbb{Q}_p, V_p E) \longrightarrow \mathbb{Q}_p \quad (\leftarrow \text{REALLY SOME } D_{dR})$$

ST

$$\exp^*(z_1) = \frac{L(E,1)}{\Omega}$$

$\Longrightarrow$ IF $L(E,1) \neq 0$, THEN $z_1 \neq 0$

3) USE DUALITY THEOREMS FROM GLOBAL GALOIS COH TO SHOW

" $\exp^*(z_1) \neq 0 \Longrightarrow \mathrm{Sel}(\mathbb{Q}, V_p E) = 0$ "

II EULER SYSTEMS

DEFN (RUBIN) LET K BE A NUMBER FIELD, V = p-ADIC REP OF G_K, ALMOST EVERYWHERE UNRAMIFIED (SAY, OUTSIDE $\Sigma \ni p$)

AN ES FOR (K,V) IS A COLLECTION OF CLASSES $(z_m)_m$

m INTEGRAL IDEAL OF K, ST $z_m \in H^1(K(m), V^*(1))$

$\uparrow$ RAY CLASS FIELD

ST.

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- z_m LANDS IN SOME FIXED LATTICE $T \subset V^*(1)$ (INDEP OF m !)

- $\mathrm{COR}^{K(m\ell)}_{K(m)}(z_{m\ell}) = \begin{cases} z_m & \text{IF } \ell \mid m \text{ OR } \ell \in \Sigma \\ P_\ell(\sigma_\ell^{-1})\, z_m & \text{ELSE} \end{cases}$ ES NORM RELN

σ_ℓ = ARITH. FROB OF ℓ, $P_\ell(X) = \det(1 - \sigma_\ell X|_V)$

RMKS - CONSIDER $V^*(1)$ B/C GLOBAL DUALITY THMS EXCHANGE V AND $V^*(1)$

- IF $z_1 \neq 0 \Rightarrow$ FINITENESS RESULTS FOR $\mathrm{SEL}(K,V)$ OR RELD SELMER GRPS

- IF $V = V_p E$, $V \cong V^*(1)$

CONJ A NONZERO ES SHOULD EXIST (MAYBE WITH $z_1 = 0$) WHENEVER V COMES FROM GEOMETRY

KNOWN CASES

1) $K = \mathbb{Q}$, $V = \mathbb{Q}_p$ CYCLOTOMIC UNITS
2) K = IM QUAD, $V = \mathbb{Q}_p$ ELLIPTIC UNITS
3) $K = \mathbb{Q}$, $V = V_p E$ OR $V = V_p f$ MOD FORM OF WT ≥ 2 KATO'S ES
[4) K = IM QUAD OR CM, $V = V_p E$ HEEGNER PTS / CYCLES]

USES RING CLASS FIELDS

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THM (LLZ, KLZ)
LET f, g BE MOD FORMS OF WT $k+2, k'+2 \geq 2$ OF LEVEL N, $p \nmid N$. LET $0 \leq j \leq k, k'$. DEFINE

$$V = V_p f \otimes V_p g\,(1+j)$$

"BEILINSON–FLACH"

THEN $\exists$ CLASSES $BF_m^{(f,g,j)} \in H^1(\mathbb{Q}(\mu_m), V^*(1))$ SATISFYING ES-LIKE RELNS AS m VARIES (BUT GOOD ENOUGH TO GET RESULTS ON SELMER GRPS)

RELN TO L-VALUES:

$$BF_1^{(f,g,j)} \in \mathrm{KER}(\exp^*)$$

BUT $\exists$ A LINEAR FNC $\log: \mathrm{KER}(\exp^*) \longrightarrow \mathbb{Q}_p$, AND ONE CAN SHOW

$$\log(BF_1^{(f,g,j)}) = (*)\, L_p(f,g,1+j)$$

HIDA'S 2-VARIABLE p-ADIC L-FN

FACT ELLIPTIC CURVE /$\mathbb{Q}$ $\longleftrightarrow$ WT 2 MOD FORM ($V_pE \cong V_p f(1)$)

2-DIM'L ODD ARTIN REP $\longleftrightarrow$ WT 1 MOD FORM

$\Longrightarrow$ DON'T GET AN ES FOR $V_p(E)(\rho)$

BUT CAN INTERPOLATE p-ADICALLY

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III A 3-VARIABLE ES

LET $\mathcal{F}, \mathcal{G}$ BE HIDA FAMILIES, $\Lambda_{\mathcal{F}}, \Lambda_{\mathcal{G}}$ THE CORRESPONDING LOCALIZATIONS OF HECKE ALGS, $M_{\mathcal{F}}, M_{\mathcal{G}}$ = Λ-ADIC REPS OF $\mathcal{F}, \mathcal{G}$, AND $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times$

$\boxed{\begin{array}{l}\text{THM (KLZ)} \\ \forall m \geq 1,\ p \nmid m,\ \exists\ BF_m^{(\mathcal{F},\mathcal{G})} \in H^1(\mathbb{Q}(\mu_m), M_{\mathcal{F}}^* \hat{\otimes} M_{\mathcal{G}}^* \hat{\otimes} \Lambda(\Gamma)) \\ \text{ST} \\ \quad \bullet \text{ THEY SATISFY ES-LIKE NORM RELNS AS } m \text{ VARIES} \\ \quad \bullet \text{ IF } f, g \text{ ARE IN } \mathcal{F}, \mathcal{G} \text{ OF WTS } k+2, k'+2 \geq 2 \text{ AND} \\ \quad\ \ 0 \leq j \leq \min(k,k'), \text{ THEN THE SPECIALIZATION OF} \\ \quad\ \ BF_m^{(\mathcal{F},\mathcal{G})} \text{ AT } (f,g,j) \text{ RECOVERS } BF_m^{(f,g,j)}\end{array}}$

- THE PTS (f,g,j) ARE DENSE IN HIDA FAMILIES

$\Longrightarrow$ GET A RELN TO p-ADIC L-VALUES EVERYWHERE IN THE FAMILIES

LET $E/\mathbb{Q}$ CORRESPOND TO f WT 2, ρ 2-DIM'L ODD ARTIN REP CORR. TO g. ASSUME f IS ORDINARY AT p. LET $\mathcal{F}, \mathcal{G}$ BE HIDA FAMILIES THROUGH f, g $\rightsquigarrow$ $BF_m^{(\mathcal{F},\mathcal{G})}$ SPECIALIZE THIS TO $(f,g,0)$ $\rightsquigarrow$ GET ES FOR $V_pE(\rho)$ RELATED TO $L(E,\rho,1)$

+ USE MACHINERY OF ES

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THM (KLZ) LET $p \geq 5$, ASSUME E DOES NOT HAVE CM, ORD AT p. LET ρ BE A 2-DIM'L ODD ARTIN REP FACTORING THROUGH F. UNDER SOME TECHNICAL HYPOTHESES (SATISFIED FOR INFINITELY MANY PRIMES),

IF $L(E, \rho, 1) \neq 0$, THEN RK $E(F)[\rho] = 0$ AND THE p-PRIMARY PART OF $Ш(E/\mathbb{Q})[\rho]$ IS FINITE

RMK. FINITENESS OF $E(F)[\rho]$ WAS FIRST PROVEN BY A DIFFERENT METHOD BY BERTOLINI - DARMON - ROTGER (CONSTRUCTED BOTTOM CLASS)

- IF ρ IS ANTICYCLOTOMIC, PROVED ~~EARLIE~~ EARLIER BY LONGO - VIGNI

- COLEMAN FAMILIES FOR NON-ORD AT p: WORK IN PROGRESS