Cohomology of locally symmetric spaces and representations of reductive groups: An introduction II.

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Bibliography

Reference for this lecture:
G. Harder, *On the cohomology of discrete arithmetically defined groups,*
J. Schwermer, *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen*, SLN 988
B. Speh, *C*ohomology of discrete groups and representation theory.

Goal for this lecture:

a.) Discuss the difference between the cohomology of compact and noncompact symmetric spaces.

b.) Explain a construction some additional harmonic forms representing nontrivial classes in $H^*(\Gamma\mathfrak{s}X)$ for noncompact locally symmetric spaces related to "cusps"

We assume that $\mathbf{G}$ is an connected algebraic group defined over $\mathbb{Q}$,
$G = \mathbf{G}(\mathbb{R})$ its real points
Γ is an arithmetic group

Recall

Theorem 1. *Suppose that $\Gamma\backslash G$ is compact.*

$$H^*(\Gamma\backslash G) = \oplus_{\pi\in\widehat{G}} m(\Gamma,\pi) H^*(\mathfrak{g}, K, V_\pi)$$

The sum is finite.

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The classification by Vogan Zuckerman can also be used for non vanishing theorem.

- If G = SU(p,q), $SU^*(2n)$, Sp(n,$\mathbb{R}$), Sp(p,q), So(p,q) and Γ an arithmetic cocompact group. A paper by J.S Li contains a list of representations π so that

$$\text{Hom}_G(\pi, L^2(\Gamma\backslash G)) \neq 0.$$

and so we get nonvanishing in the corresponding degrees.

On the other hand geometric results about Euler characteristics imply non vanishing theorems

- If $G = Sp(2n, \mathbb{R})$, $SU(p, q$, $SO(2n, 2m))$, then $H^i(\Gamma \mathfrak{s} G)$ has nontrivial cohomology in the middle degree. (Clozel, Rohlfs-S, Savin)

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As unitary representation of G

$$L^2(\Gamma\backslash G) = \tilde{\bigoplus} m_{L^2}(\Gamma,\pi)V_\pi \oplus L^2_{con}(\Gamma \mathfrak{s} G)$$

In our example:

$G = Sl(2,\mathbb{R})$, Γ subgroup of $Sl(2,\mathbb{Z})$.
$\Gamma\backslash X$ has finite volume, but it is not compact.
In analytic number theory it is proved that classical cusp form are in $L^2(\Gamma\backslash G)$

We have a choice of cohomology theories. We can consider

- $H^*_{L^2}(\Gamma\backslash X)$ L^2–cohomology

- $H^*_2(\Gamma\backslash X)$ cohomology represented by harmonic square integrable forms.

- $H^*_{deRham}(\Gamma\backslash X)$

All give interesting information and results

Problem: $H_{L^2}(\Gamma\backslash X)$ maybe infinite dimensional. But if not

Theorem 2. *(Borel +......)*

Suppose dim $H_{L^2}(\Gamma\backslash X) < \infty$ *then*

$$H_{L^2}(\Gamma\backslash X) = \oplus_{\pi\in\widehat{G}} m_{L^2}(\Gamma,\pi) H^*(\mathfrak{g},K,\pi).$$

The sum is finite (and may be restricted to those unitary representations $\pi \in \widehat{G}$ *with trivial infinitesimal character).*

Warning: There is map

$$H^*_{L^2}(\Gamma\backslash X) \to H^*(\Gamma\backslash X)$$

but this map is **not** injective. It is an open problem to determine the kernel.
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In our example:
The constant function is square integrable
Thus the trivial representation Id is a direct summand of $L^2(\Gamma\backslash G)$
$H^2(\mathfrak{g}, K, Id) = \mathbb{C}$ since the volume form on G/K is right invariant under G.
But $\Gamma\backslash X$ is not compact and hence $H^2(\Gamma\backslash X) = 0$.

Important Theorems

Let $\mathcal{A}(\Gamma\backslash G)$ be the space of automorphic forms on $\Gamma\backslash G$

In our example: The automorphic forms on the upper half plane (holomorphic cusp forms, Eisenstein series, Maass forms) lift to automorphic functions on $\Gamma\backslash Sl(2,\mathbb{R})$

Theorem 3. *(Franke)*

$$H^*_{deRham}(\Gamma\backslash X) = H^*(\mathfrak{g}, K, \mathcal{A}(\Gamma\backslash G))$$

In other words :

$$H^*_{deRham}(\Gamma\backslash X)$$

is related number theory and to automorphic forms.

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but as we will see it is complicated to understand in detail.

Towards the cuspidal contribution to cohomology

In our example: Recall that the cusp forms on the upper half plane (Holomorphic and anti holomorphic cusp forms, and Maass forms lift to square integrable functions on $\Gamma\backslash Sl(2,\mathbb{R})$. The closure of this space is denoted by

$$L^2_{cusp}(\Gamma\backslash SL(2,\mathbb{R})) \subset L^2(\Gamma\backslash SL(2,\mathbb{R})).$$

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By a theorem of Piatetskii-Shapiro and Gelfand

$$L^2_{cusp}(\Gamma\backslash G) = \oplus_{\pi\in\widehat{G}_u} m_{cusp}(\pi,\Gamma)\pi.$$

Let

$$J : \mathrm{Hom}_K(\wedge^* p, C^\infty(\Gamma\backslash G) \cap L^2_{cusp}(\Gamma\backslash G)) \to \mathrm{Hom}_K(\wedge^* p, C^\infty(\Gamma\backslash G)).$$

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Theorem 4. *(Borel)*

$$J^* : H^*(\mathfrak{g}, K, C^\infty_{cusp}(\Gamma\backslash G)) \to H^*(\mathfrak{g}, K, C^\infty(\Gamma\backslash G))$$

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$$J^* : H^*(\mathfrak{g}, K, C^\infty_{cusp}(\Gamma\backslash G)) \to H^*(\mathfrak{g}, K, C^\infty(\Gamma\backslash G))$$

is injective.

The image of J^* is the cuspidal cohomology

$$H^*_{cusp}(\Gamma\backslash X) \cong \oplus_{\pi \in \widehat{G}_u} m_{cusp}(\pi, \Gamma) H^*(\mathfrak{g}, K, \pi).$$

In our example: $G = Sl(2,\mathbb{R})$

From the last lecture :
π is an irreducible infinite dimensional representation with $H^*(\mathfrak{g}, K, \pi) \neq 0$ implies π has cohomology in degree 1.

This representation π is generated by the lift to $\Gamma\backslash Sl(2,\mathbb{R})$ of a classical holomorphic or anti holomorphic cusp form .

$m_{cusp}(\pi, \Gamma)$ is the dimension al the space of classical holomorphic cusp forms of weight 2 on $\Gamma\backslash H$.

Note: The cohomology doesn't give us any information about the Maass forms.

Towards understanding the part of the cohomology contributed by the cusps of $\Gamma\backslash G$

We consider the Borel Serre compactification $\overline{\Gamma\backslash X}$ of $\Gamma\backslash X$

In our example : we compactly $\Gamma\backslash X$ by adding a circle at infinity at the cusp.

Borel-Serre proved:

$$H^*(\overline{\Gamma\backslash X}, \mathbb{C}) = H^*_{deRham}(\Gamma\backslash X, \mathbb{C})$$

Consider the restriction to the boundary

$$\mathrm{Res}^* : H^*(\overline{\Gamma\backslash X}, E) \to H^*(\partial\overline{\Gamma\backslash X}, E)$$

$H^*_!(\Gamma\backslash X, E)$ be the kernel of Res.

The cohomology classes which have a nontrivial restriction to a face of the Borel Serre compactification are called Eisenstein cohomology classes.

In our example: $G = Sl(2, \mathbb{R})$

$$H^*_{deRham}(\Gamma\backslash X, \mathbb{C}) = H^*_!(\Gamma\backslash X, \mathbb{C}) \oplus H^*_{Eis}(\Gamma\backslash X, \mathbb{C})$$

and

$$H^*_!(\Gamma\backslash X, \mathbb{C}) = H^*_{cusp}(\Gamma\backslash X, \mathbb{C})$$

Caution: Unfortunately both of these statement are not true in general.

To understand the Eisenstein classes we have to get some understanding of the Borel Serre compactification and use it to construct a subspace of the space of automorphic forms.

Assume that Γ is a congruence subgroup and $\Gamma\backslash G$ not compact.

Thus there exist nontrivial *parabolic subgroups P* defined over $\mathbb{Q}$, P can be written as a product of a Levi subgroup L and a unipotent radical U

We can write $b \in B$ is of the form $b = lu$ with $l \in L$, $u \in U$. We can write L= MA where are is the maximal abelian connected subgroup in the center of L

In our example: G= Sl(2,$\mathbb{R}$),
The subgroup B of upper triangular matrices is a parabolic subgroup.
Let L be the diagonal matrices in B.
U the subgroup of B with diagonal entries (1,1).

L is the Levi subgroup of B and U the unipotent radical. A are the diagonal matrices in L with positive entries $M = +/-I$

The Borel Serre compactification of $\Gamma \backslash X$ is obtained as follows: For each rational parabolic P, choose a boundary face $e(P)$ attach the boundary face e(P) at ∞ at X and form the completion $X \cup \bigcup_P e(P)$. Γ acts continuously on it. The quotient $\Gamma \backslash X \cup \bigcup_P e(P)$ is the Borel Serre compactification $\overline{\Gamma \backslash X}$ with boundary $\partial \overline{\Gamma \backslash X}$.

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In our example: $G = SL(2, \mathbb{R})$ Rational parabolic subgroups correspond to rational points in the boundary of X and $e(P) = \Gamma \cap U\backslash U$.

Construction of submodules of $\mathcal{A}(\Gamma, G)$

For a rational parabolic subgroup $P = LN$ we define $\Gamma_P = \Gamma \cap P$, and $\Gamma_L = \Gamma \cap L$ and let π_L is an irreducible subrepresentation of of $L^2_{cusp}(\Gamma_L \backslash L)$.
Consider the space

$$I(P, \pi_L,) = \{f \in C^\infty(\Gamma_P A_L N \backslash G, V_{\pi_L}) | f(lg) = \pi_L(l) f(g) \text{ for } g \in G, l \in L\}$$

We define for a character μ of A an action of G on

$$I(P, \pi_L, \mu)(g) : I(P, \pi_L,) \to I(P, \pi_L,)$$

We define the Eisenstein intertwining operator

$$E(P, \pi_L, \mu) : I(P, \pi_M, \mu) \to \mathcal{A}(\Gamma \backslash G) \subset C^{\infty}(\Gamma \backslash G)$$

by

$$(E(P, \pi_L, \mu) f)(g) = \sum_{\gamma \in \Gamma_p \backslash \Gamma} f(\gamma g)$$

for $f \in I(P, \pi_L, \mu)$,. This operator can be continued to a meromorphic function of μ with singularities on a finite number of hyperplanes.

In our example: $G = Sl(2, \mathbb{R})$
After unwinding all the definitions we see $(E(P, \pi_L, \mu) f)(g)$ is an Eisenstein series.

Note We may replace this Eisenstein intertwining operator by an normalized operator to take care of some of the convergence problems.

Then

$$E(P,\pi_L,\mu)^* : H^*(\mathfrak{g},K,I(P,\pi_L,\mu)\otimes E) \\ \rightarrow\ H^*(\mathfrak{g},K,A(\Gamma\backslash G))\ =\ H^*(\Gamma\backslash X)$$

The nonzero classes in the image of $E(P,\pi_L,\mu)^*$ for a rational subgroup P, a dominant character μ of L and a representation π_L in $L^2(\Gamma_L\backslash L/A)$ are called **Eisenstein classes**.

Their restriction to the face $e(P)$ of the Borel Serre compactification is nonzero.

Computing the Eisenstein cohomology

If the operator Eisenstein operator is injective $H^*(\mathfrak{g}, K, I(P, \pi_L, \mu))$ and it image is know.

If the Eisenstein operator is not injective then except for groups of rank one and some small groups no complete results are known.

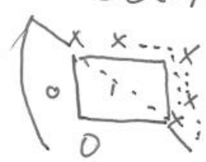
Title: Cohomology of Locally Symmetric Spaces and Representations of Reductive Groups: An Introduction II

Speaker: Birgit Speh

Date: 2014.08.19 Time: 02:00pm

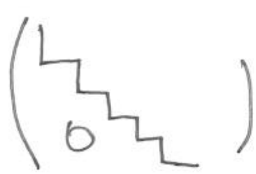
$P \subset SL(n,\mathbb{R})$

$SO(n,1)$



$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$



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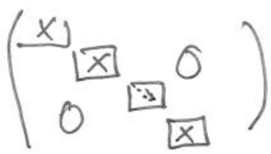

$$\begin{pmatrix} a & u \\ 0 & a^{-1} \end{pmatrix}$$

$$e(B) = \begin{pmatrix} 1 & u/mu \\ & 1 \end{pmatrix}$$