Title: Intro to p-adic Galois Representations

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Intro to p-adic Galois Representations

Tate: recall $A/_{\mathbb{C}}$ ab. var.

$$A = \mathbb{C}^g/\Lambda \qquad \mathrm{rk}_{\mathbb{Z}} \Lambda = 2g$$

$$H_1(A, \mathbb{Z}) = \Lambda$$

$$H^1(A, \mathbb{C}) = \mathrm{Hom}(\Lambda, \mathbb{C}) \simeq H^1_{dR}(A)$$

$$\left(\gamma \mapsto \int_\gamma \omega\right) \leftarrow\!\shortmid \omega$$

$$H^1_{dR}(A) = H^0(A, \Omega^1_A) \oplus H^1(A, \mathcal{O}_A)$$

Note: $H_1(A, \mathbb{Z}/p^n\mathbb{Z}) = \frac{1}{p^n}\Lambda/\Lambda$

$$= A[p^n]$$

$$H_1^{\acute{e}t}(A, \mathbb{Z}_p) = \varprojlim A[p^n] = T_p A$$

<u>Thm</u> (Tate) Let $K/\mathbb{Q}_p$ finite, $A/_K$ ab. var. good reduction. Let $C = \widehat{\bar{K}}$

$$H^1(A_{\bar{K}}, \mathbb{Z}_p) \otimes C \simeq H^0(A, \Omega^1_{A|C})(-1)$$
$$\oplus H^1(A_C, \mathcal{O}_{A_C}) \simeq C^g(-1) \oplus C^g$$

as C v.s. w/ semi linear G_K-action

(Hodge-Tate decomp.)

$G_K \circlearrowright C$ $\qquad$ V C-v.s.

$\sigma : V \to V \quad \forall \sigma \in G_K$

$\sigma(\alpha v) = \sigma(\alpha)\sigma(v), \ \alpha \in C$

$\chi_{cycl} : G_K \to \mathbb{Z}_p^*$

$\sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{cycl}(\sigma)}$

$C(n) = Ce$
$\sigma(e) = \chi_{cycl}^n(\sigma)e$

<u>Ex</u>: $E/_{\mathbb{Q}_p}$ good ordinary reduction

$$0 \to \varprojlim \hat{E}[p^n](\overline{\mathbb{Z}}_p) \to \underbrace{\varprojlim E[p^n](\overline{\mathbb{Z}}_p)}_{\textstyle\overset{T_pE}{\shortparallel}} \to \varprojlim E[p^n](\overline{\mathbb{F}}_p) \to 0$$

$T_pE \otimes C$, action of $G_{\mathbb{Q}_p}$ is $\begin{pmatrix} \chi\delta^{-1} & * \\ 0 & \delta \end{pmatrix}$

$* \in H^1(G_{\mathbb{Q}_p}, C(1) \otimes \delta^2)$

<u>Thm</u>: Let $\eta : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^*$ be a char, $\eta(I_{\mathbb{Q}_p})$ infinite, then

$$H^i(G_{\mathbb{Q}_p}, C(\eta)) = 0 \ , \quad i = 0, 1.$$

Proof involves study of tower K_∞ / K

$$\begin{array}{cl} \bar{K} & \\ \mid & \\ K_\infty & \\ \mid & \Gamma \\ K & \end{array}$$

Assume $K_\infty = \bigcup_{n \geq 0} K_n$

For $n \gg 0$ $\quad K_{m+n} / K_n$

cyclic deg p^m and tot. ramified, <u>e.g.</u> $K_n = K(\mu_{p^n})$

<u>Prop</u> $\exists\, 0 < \varepsilon < 1$ s.t. $\forall\, n \gg 0$

$\forall g \in \mathrm{Gal}(K_{n+1}/K_n) \quad \forall x \in \mathcal{O}_{K_{n+1}}$,

$$g(x) \equiv x \pmod{p^\varepsilon}$$

In case, $K_n = \mathbb{Q}_p(\mu_{p^n})$

$$\mathrm{Gal}(K_{n+1}/K_n) \simeq \frac{1+p^n\mathbb{Z}_p}{1+p^{n+1}\mathbb{Z}_p}$$

$$g(\zeta_{p^{n+1}}) - \zeta_{p^{n+1}} = \zeta_{p^{n+1}}(\zeta_p^a - 1) \equiv \mathcal{O}(p^{\frac{1}{p-1}}) \leftarrow 1+p^n a$$

Cor: $\exists\, 0<\varepsilon<1$, $\forall\, n \gg 0$ $\forall\, x \in \mathcal{O}_{K_{n+1}}$,

$$N_{K_{n+1}/K_n}(x) \equiv x^p \, (p^{\varepsilon})$$

Now define some rings of char p.

perfect norm field $\quad \tilde{E}_K^+ = \varprojlim_{\mathrm{Frob}_p} \mathcal{O}_{K_\infty}/p^{\varepsilon}$

$\cup \qquad\qquad = \{(x_0, x_1, \ldots) : x_i^p = x_{i-1}\}$

$E_K^+ = \{ x \in \tilde{E}_K^+ : \forall\, n \gg 0,\ x_n \in \mathcal{O}_{K_n}/p^{\varepsilon} \}$

let k_n = res field of K_n

$k_\infty = \cup\, k_n = k_N$ for $N \gg 0$

then $k_n \hookrightarrow \mathcal{O}_{K_n}/p^{\varepsilon}$ via Teich. lift

Get $k_\infty \hookrightarrow E_K^+$

cyclotomic case $K_n = \mathbb{Q}_p(\mu_{p^n})$

E_K^+ contains

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in E_K^+$$

$$T = \varepsilon - 1 = (0, \zeta_p - 1, \zeta_{p^2} - 1, \dots)$$

uniformizer

T is topologically nilpotent

Easy to check: $\mathbb{F}_p[[T]] \xrightarrow{\sim} E_K^+$

<u>Prop</u>: If $n >> 0$, $\pi_n \in \mathcal{O}_{k_n}$ is a uniformizer

then $\exists\, \pi_{n+1} \in \mathcal{O}_{k_{n+1}}$, $\pi_{n+1}^p \equiv \pi_n \bmod p^\varepsilon$

<u>Pf</u>: Let $\varpi_{n+1} \in \mathcal{O}_{K_{n+1}}$ be a unif.

then $N(\varpi_{n+1}) = \sum_{i=1}^{\infty} [a_i] \pi_n$ $\quad a_i \in k_n$

unif
for K_n

want to find $\pi_{n+1} = \sum [b_i] \varpi_{n+1}^i$

so that $\pi_{n+1}^p \equiv \pi_n \ (p^\varepsilon)$

$$\sum [b_i^{\,p}]\, \varpi_{n+1}^{p^i} \equiv \pi_n$$
$$\underset{N\varpi_{n+1}^{i}}{\shortparallel}$$

Solve for b_i's.

Let $\pi = (\ldots, \pi_n, \pi_{n+1}, \ldots) \in E_K^+$

$\uparrow \nearrow \nearrow \ldots$

unifs

<u>Thm</u>: a) $E_K^+ \simeq k_\infty[[X]]$ (let $E_K = k_\infty((X))$)

$\pi \longleftarrow\!\!| \; X$

b) $\tilde{E}_K^+ \simeq k_\infty[[X^{\frac{1}{p^\infty}}]]$ (let $\tilde{E}_K = k_\infty((X^{\frac{1}{p^\infty}}))$)

Have a map

$$\theta : W(\tilde{E}_K^+) \longrightarrow \mathcal{O}_{\hat{K}_\infty}$$
$$[x] \longmapsto \lim_{n\to\infty} \tilde{x}_n^{\,p^n}$$

$x = (x_0, x_1, \ldots) \quad x_n \in \mathcal{O}_{K_\infty}/p^{\varepsilon}$

$\tilde{x}_0, \tilde{x}_1, \ldots \in \mathcal{O}_{K_\infty}$

ker is (ω)

In cyclotomic case

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \tilde{E}_K$$

$$\theta([\varepsilon]) = 1$$
$$\theta([\varepsilon^{\frac{1}{p}}]) \equiv \zeta_p \qquad\qquad \omega = \frac{[\varepsilon]-1}{[\varepsilon^{\frac{1}{p}}]-1}$$

If M/K_∞ is a finite extn.

$$M = \bigcup M_n \qquad M_n = M_0 K_n, \quad n \gg 0$$

$\rightsquigarrow E_M/E_K$ finite sep ext'n, $E_K \simeq k_\infty((X))$

<u>Thm</u>: (Fontaine Wintenberger) $\exists$ equivalences

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{fin extns} \\ \text{of } K_\infty\end{array}\right\} \arrow[r, "\sim", "M \mapsto E_M"'] \arrow[d] & \left\{\begin{array}{c}\text{fin sep extns} \\ \text{of } E_K\end{array}\right\} \arrow[d, "\wr"] \\
\left\{\begin{array}{c}\text{fin extns} \\ \text{of } \hat{K}_\infty\end{array}\right\} \arrow[r, "\sim"] & \left\{\begin{array}{c}\text{fin sep. extns} \\ \text{of } \tilde{E}_K\end{array}\right\}
\end{tikzcd}$$

$$W(\mathcal{O}_L) \otimes_{W(\tilde{E}_K^+), \theta} \hat{K}_\infty \longleftarrow\!\shortmid L$$

$$\|$$

$$W(\mathcal{O}_L)[\tfrac{1}{p}]/\omega$$

<u>Cor</u>

$$G_{K_\infty} \simeq G_{E_K}$$

$$\begin{tikzcd} \bar{K} \arrow[d, dash] \\ K_\infty \arrow[d, dash, "\Gamma"'] \\ K \end{tikzcd}$$

Γ acts on $E_K \simeq k_\infty((X))$

In cycl. case $\Gamma \simeq \mathbb{Z}_p^*$

$$\gamma_a \mapsfrom a$$

$$\gamma_a(X) = (1+X)^a - 1$$