Title: phi-Gamma modules and p-adic Hodge theory

Speaker:

Date: 2014. 08. 22 Time: 03:30pm

- In the example, from yesterday, you have to work with $V_p E$ instead of $T_p E$ $(\mathbb{Q}_p, \mathbb{Z}_p)$

- Herr complex

$$0 \to D \to D \oplus D \to D \to 0$$

$$x \mapsto ((\varphi-1)x, (\gamma-1)x)$$

$$(x,y) \mapsto (\gamma-1)x - (\varphi-1)y$$

- $\sum_{n\geq 0} \varphi^n(f)$ converges in $\mathbb{Z}_p[[T]]$ if $f \in T\mathbb{Z}_p[[T]]$ for (p,T)-adic topology

$$\varphi^n(T) = (1+T)^{p^n} - 1$$

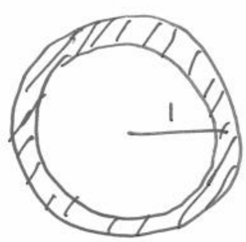
$\mathbb{Z}_p$-reps of $G_{\mathbb{Q}_p} \xleftrightarrow{\sim} \varphi\Gamma^{et}(\mathcal{O}_{\mathcal{E}})$

$$\mathcal{O}_{\mathcal{E}} = \Big\{ \sum_{n\in\mathbb{Z}} a_n T^n \;\Big|\; a_n \in \mathbb{Z}_p \quad \begin{matrix} a_n \to 0 \\ n \to -\infty \end{matrix} \Big\}$$

$$\mathcal{E} = \mathcal{O}_{\mathcal{E}}[\tfrac{1}{p}] = \mathrm{Frac}\, \mathcal{O}_{\mathcal{E}}$$

$\mathbb{Q}_p$-reps of $G_{\mathbb{Q}_p} \longleftrightarrow \varphi\Gamma^{et}(\mathcal{E})$

$$\varphi\Gamma^{et}(\mathcal{E}) = \{ \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} D \mid D \in \varphi\Gamma^{et}(\mathcal{O}_{\mathcal{E}}) \}$$



$\mathcal{R}$ = Robba ring $\quad \boxed{p^{-\frac{1}{p^{n-1}(p-1)}}} \leftarrow R_n$

$$= \varinjlim \underbrace{\mathcal{E}^{]0, R_n]}}$$

$0 < v_p(x) \le R_n$

analytic functions on the annulus $|\zeta_{p^n} - 1| \le |X| < 1$

$$t = \log(1+T) = \sum_{n\ge 1} (-1)^{n-1} \frac{T^n}{n} \in \mathcal{R}$$

Exercise: $t\mathcal{R}$ is dense in $\mathcal{R}$

$\varphi, \Gamma \curvearrowright \mathcal{R} \qquad \varphi(T) = (1+T)^p - 1$

$\qquad\qquad\qquad \gamma(T) = (1+T)^{\chi(\gamma)} - 1$

Γ preserves each $\mathcal{E}^{]0,R_n]}$ but

$$\varphi(\mathcal{E}^{]0,R_n]}) \subset \mathcal{E}^{]0,R_{n+1}]}$$

Definition: $\varphi\Gamma(\mathcal{R})$ = finite free $\mathcal{R}$-modules

$\varphi\Gamma$-modules over $\mathcal{R}$ $\quad D + \varphi : D \to D$ φ-semilinear

$$\mathcal{R} \otimes_{\varphi,\mathcal{R}} D \simeq D$$

+ commuting C^0 action of Γ

Example: $\delta : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times \rightsquigarrow$

define $\mathcal{R}(\delta) = \mathcal{R} \cdot e \quad \varphi(e) = \delta(p)e$

$$\gamma(e) = \delta(\chi(\gamma))e$$

Prop: $\forall D \in \varphi\Gamma(\mathcal{R})$ $\exists n(D)$ and for $n \geq n(D)$ some canonical $\mathcal{E}^{]0,R_n]}$-submodule $D^{]0,R_n]} \subset D$ such that

- $\mathcal{R} \otimes D^{]0,R_n]} \xrightarrow{\sim} D$
- $D^{]0,R_n]}$ is stable under Γ

- $\varphi\left(D^{]0,R_n]}\right) \subset D^{]0,R_{n+1}]}$

$\mathcal{E} \rightsquigarrow \mathcal{E}^\dagger \leftsquigarrow \mathcal{R}$

$\mathcal{E}^\dagger$ ← overconvergent elements "objects in $\mathbb{Q}_p[[T,\frac{1}{T}]]$ which are in $\mathcal{E}$ and $\mathcal{R}$

$\mathcal{E}^\dagger \to$ bounded functions in $\mathcal{R}$

<u>Theorem</u>: (Cherbonnier - Colmez)

$\forall\, D \in \varphi\Gamma^{et}(\mathcal{E})$ there is a largest $\mathcal{E}^\dagger$ vector space $D^\dagger \subset D$, finite dimensional, Stable under φ, Γ, and $\mathcal{E} \otimes_{\mathcal{E}^\dagger} D^\dagger \longrightarrow D$ is an isomorphism.

You get a functor

$$\mathbb{Q}_p\text{-reps of } G_{\mathbb{Q}_p} \longrightarrow \varphi\Gamma(\mathcal{R})$$
$$V \longmapsto \mathcal{R} \otimes_{\mathcal{E}^\dagger} D(V)^\dagger$$

$V \mapsto D(V)^\dagger$

<u>Thm</u>: (Kedlaya)

$$\mathbb{Q}_p\text{-reps of } G_{\mathbb{Q}_p} \simeq \varphi\Gamma^{et}(\mathcal{R})$$

Recover D_{cris}?

V <u>any</u> p-adic rep of $G_{\mathbb{Q}_p}$

$\rightsquigarrow D_{rig}(V) = \mathcal{R} \otimes_{\mathcal{E}^+} D^+(V)$

<u>Thm</u>: (Berger)

$$\underset{\varphi\text{-module}/\mathbb{Q}_p}{D_{cris}(V)} \cong \left(D_{rig}(V)[\tfrac{1}{t}]\right)^{\Gamma}$$

$$= \text{in } \tilde{B}^+_{rig}[\tfrac{1}{t}] \otimes V$$

Moreover, if V is crystalline,

$$\mathcal{R}[\tfrac{1}{t}] \otimes_{\mathbb{Q}_p} D_{cris} \simeq \underbrace{\mathcal{R}[\tfrac{1}{t}] \otimes_{\mathcal{R}} D_{rig}(V)}_{D_{rig}(V)[\frac{1}{t}]}$$

What about D_{dR}?

Recall $1, \zeta_p, \zeta_{p^2}, \ldots$

$f \in \mathcal{E}^{]0, R_n]}$ $\quad f(\zeta_{p^n} - 1) \in \mathbb{Q}_p(\mu_{p^n}) = F_n$ makes sense

Can define $\varphi^{-n}(f) = f(\zeta_{p^n} e^{\frac{t}{p^n}} - 1) \in F_n[[t]] \subset B_{dR}^+$

Get a map

$\overline{\mathbb{Q}}_p \subset B_{dR}^+$ so $F_n \subset B_{dR}^+$

$$\begin{tikzcd} \Sigma^{]0,R_n]} \arrow[r, hook, "C^0"] \arrow[dr] & B_{dR}^+ \\ & F_n[[t]] \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$G_{\mathbb{Q}_p}$ acts on LHS via Γ

$t = \log[\varepsilon] \in B_{dR}^+$

In general one can define for any $D \in \varphi\Gamma^{et}(\mathcal{R})$ a C^0 embedding

$$D^{]0,R_n]} \hookrightarrow B_{dR}^+ \otimes V$$

where V is such that $D_{rig}(V) = D$.

Define

$$\begin{array}{c} D^+_{dif,n} = F_n[[t]] \otimes_{\varphi^{-n}, \Sigma^{]0,R_n]}} D^{]0,R_n]} \\ \downarrow \\ B_{dR}^+ \otimes V \end{array}$$

is still injective

$\Gamma \curvearrowright D^+_{dif,n}$

<u>Thm</u> (Fontaine, Cherbonnier-Colmez

Assume V is de Rham

then $\forall i \in \mathbb{Z}$

$$\mathrm{Fil}^i D_{dR}(V) = (t^i D^+_{dif,n})^\Gamma \quad \forall n \gg 0$$

Reconstruction of D_{rig}

Suppose V is crystalline

$$\mathcal{R}[\tfrac{1}{t}] \otimes D_{cris} \xrightarrow{\sim} \mathcal{R}[\tfrac{1}{t}] \otimes D_{rig}$$

How do you recover $\mathcal{R}$ from $\mathcal{R}[\tfrac{1}{t}]$?

<u>Prop</u>: $f \in \mathcal{R}$, then

$$f(\zeta_{p^n} - 1) = 0 \quad \forall n \gg 0 \iff t \mid f \text{ in } \mathcal{R}.$$

$$\underbrace{\varphi^{-n}(f)}_{f(\zeta_{p^n} e^{t/p^n} - 1)} \bmod t$$

$$\varphi^{-n}(f) \in t \cdot F_n[[t]] \quad \forall n \gg 0$$

<u>Thm</u>: $D_{rig} = \{ z \in D_{rig}[\frac{1}{t}] \mid \varphi^{-n}(z) \in D^{+}_{dif,n} \quad \forall n \gg 0 \}$

If V crystalline we get

$$\boxed{D_{rig} = \{ z \in \mathcal{R}[\tfrac{1}{t}] \otimes D_{cris} \mid \varphi^{-n}(z) \in \mathrm{Fil}^0(F_n[[t]] \otimes D_{cris}) \quad \forall n \gg 0 \}}$$

<u>Thm</u> (Berger) The functor

$$\mathcal{D} \to \underbrace{\{ z \in \mathcal{R}[\tfrac{1}{t}] \otimes \mathcal{D} \mid \varphi^{-n}(z) \in \mathrm{Fil}^0(F_n[[t]] \otimes \mathcal{D}) \quad \forall n \gg 0 \}}_{= \mathcal{M}(\mathcal{D})}$$

is exact, fully faithful, $\otimes$ functor from filtered φ-mod to $\varphi\Gamma(\mathcal{R})$, and $\mathcal{D}$ is weakly admissible $\Longleftrightarrow$ $\mathcal{M}(\mathcal{D})$ is étale.