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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Peter Scholze

Talk Title: Integral p-adic Hodge Theory

Date: 12 / 1 / 2014 **Time:** 2 : 00 am / [✓ pm] (circle one)

List 6-12 key words for the talk: Integral p-adic Hodge Theory, Breuil-Kisin modules, integral comparison theorem, topological Hoschild homology

Please summarize the lecture in 5 or fewer sentences: Describes new results comparing integral de Rham, étale, and crystalline cohomologies of rigid analytic varieties and formal schemes. The main tool is a cohomology theory with values in Breuil-Kisin modules. The construction is motivated by a trip to "outer space" with topological Hochschild homology, though it is not necessary for the proof of the results.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Integral p-adic Hodge Theory
(Work in progress w/Bhargav Bhatt, Matthew Morrow)

Reality **Problem** $K/\mathbb{Q}_p$ finite, $\mathcal{X}/\mathcal{O}_K$ a proper smooth formal scheme

Relate $H^i_{cris}(\mathcal{X}_s/W)$

and $H^i_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$

integrally

IF $K/\mathbb{Q}_p$ is unramified (or close) and i is small, Fontaine-Lafaille theory.

Let $C/\mathbb{Q}_p$ be alg. closed and complete, k residue field

Let $A_{inf} = W(\mathcal{O}_{C^\flat})$ $\left(\mathcal{O}_{C^\flat} = \varprojlim_{\Phi} \mathcal{O}_C/p\right)$

$$\begin{tikzcd} A_{inf} \arrow[loop left, "\varphi"] \arrow[d, "\theta"] \\ \mathcal{O}_C \end{tikzcd}$$

$\operatorname{Ker}\theta$ generated by $\xi = \dfrac{[\varepsilon^p]-1}{[\varepsilon]-1}$

$\varepsilon = (\zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_{C^\flat}$

Def'n (Fargues) A Breuil-Kisin module / A_{inf} is a finite free A_{inf}-module M with
$\varphi_M : (\varphi^* M)[\xi^{-1}] \xrightarrow{\sim} M[\xi^{-1}]$

Example $\underbrace{A_{inf}(-1)}_{\text{Tate twist in B-K world}} := A_{inf}$ $\varphi_{A_{inf}(-1)} = \xi \cdot \varphi_{A_{inf}}$

(So have fixed a compatible system of roots of unity)

Remark A B-K module has several realizations:

1) étale: $T(M) := (M \otimes_{A_{\inf}} W(C^\flat))^{\varphi=1}$ finite free $\mathbb{Z}_p$-module

Then $M[\frac{1}{\nu}] \cong T(M) \otimes_{\mathbb{Z}_p} A_{\inf}[\frac{1}{\nu}]$

$$\nu = [\varepsilon]\cdot 1 = \text{``}\varphi^{-1}(\xi)\,\varphi^{-2}(\xi)\cdots\text{''}$$

2) de Rham: $(\varphi^* M[\frac{1}{p}])^{\wedge}_{\xi}$ ← completion

and

$$M[\tfrac{1}{p}]^{\wedge}_{\xi} = T(M) \otimes_{\mathbb{Z}_p} B^+_{dR}$$

$$B^+_{dR} = A_{\inf}[\tfrac{1}{p}]^{\wedge}_{\xi}$$

two B^+_{dR}-lattices in same B_{dR}-VS.

$$\Xi := (\varphi^* M[\tfrac{1}{p}])^{\wedge}_{\xi} \subseteq T(M) \otimes_{\mathbb{Z}_p} B_{dR}$$

is a B^+_{dR}-lattice

3) crystalline: $M \otimes_{A_{\inf}} W(k)$ $\circlearrowleft \varphi_M$

F-crystal / k

Theorem (Fargues) The functor

$$\{\text{B-K modules}/A_{\inf}\} \to \{(T, \Xi) \mid T \text{ a fin. free } \mathbb{Z}_p\text{-mod}, \ \Xi \subseteq T \otimes B_{dR} \text{ a } B^+_{dR}\text{-lattice}\}$$

is an equivalence of categories

Theorem (S.+ Conrad-Gabber)

Let X/C be a proper smooth rigid analytic variety

Then $H^i_{ét}(X, \mathbb{Z}_p)$ is a f.g. $\mathbb{Z}_p$-mod, and there is a natural B_{dR}^+-lattice $\Xi \subseteq H^i_{ét}(X, \mathbb{Z}_p) \otimes B_{dR}$

s.t.
$$\Xi \otimes_{B_{dR}^+} C = H^i_{dR}(X)$$

If $X = X_K \hat{\otimes}_K C$, $K/\mathbb{Q}_p$ finite,

then $\Xi = H^i_{dR}(X_K) \otimes_K B_{dR}^+$

Cor For any X/C, get a B-K module

$H^i_{A_{inf}}(X)$ associated w/ $H^i(\mathbb{Z}_p)/\text{torsion}$, Ξ

Theorem (Bhatt-Morrow-S.) Let $\mathfrak{X}/\mathcal{O}_C$ be a proper smooth formal scheme. Then,

1) If $H^i_{crys}(\mathfrak{X}_k/W(k))$ is torsion-free, then $H^i_{ét}(\mathfrak{X}_C, \mathbb{Z}_p)$ is torsion-free

2) If H^i_{crys}, H^{i+1}_{crys} are torsion-free, then

$$\circledast \; H^i_{crys}(\mathfrak{X}_k/W(k)) \cong H^i_{A_{inf}}(X) \otimes_{A_{inf}} W(k)$$

Remark

1) New for K3 surfaces over 2-adic fields

2) If $\mathfrak{X}$ defined $/\mathcal{O}_K$, $K/\mathbb{Q}_p$ finite, then $H^i_{A_{inf}}$ can be recovered from $H^i_{ét} \circlearrowleft G_K$

Strategy Define a new cohomology theory $R\Gamma_{A_{inf}}(\mathfrak{X})$: perfect complex of A_{inf}-modules + φ-action which compares well w/all other cohomology theories

Outer space Topological Hochschild homology

Given (usual) ring A, there is a "cyclotomic spectrum"

$THH(A) \circlearrowleft S^1$

For each fin. subgroup $C \subseteq S^1$ have "genuine C-fixed point spectrum"

$$TR^C(A) := THH(A)^C$$

If $C = C_{p^{r-1}}$, write $TR^r(A;p) = TR^{C_{p^{r-1}}}(A)$

There are maps
$$\begin{tikzcd} TR^{r+1}(A;p) \arrow[r, shift left, "F"] \arrow[r, bend left=40, "R"] & TR^r(A;p) \arrow[l, shift left, "V"] \end{tikzcd}$$

F = inclusion of fixed points "Frobenius"

V = "trace" = "Verschiebung"

$R \sim$ something to do w/cyclotomic spectrum

Thm (Hesselholt-Madsen) $\pi_0 TR^r(A;p) = W_r(A)$ compatible w/F, R, V

Let $TF^\bullet(A;p)$ = system of $TR^r(A;p)$ w/F as transition map

$$\left[A_{inf} = \varprojlim_F W_r(\mathcal{O}_C) \right]$$

$TF(A;p) := \varprojlim_F TF^\bullet(A;p)$ $\quad$ $TF(A;p,\mathbb{Z}_p)$ = its p-adic completion "Topological Frobenius homology"

Thm (Hesselholt)

$$\pi_i \underset{R \circlearrowright}{TF}(\mathcal{O}_{\mathbb{C}_p}; p, \mathbb{Z}_p) = \begin{cases} \overset{\circlearrowright \varphi^{-1}}{A_{\inf}(i/2)} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Conj There is an E_2-spectral sequence

$$\underset{\circlearrowleft_{\varphi^{-1}}}{H^i_{A_{\inf}}(\mathfrak{X})(-j/2)} \Rightarrow \pi_{-i-j} \underset{R \circlearrowleft}{TF}(\mathfrak{X}; p, \mathbb{Z}_p)$$

($i \geq 0$, $j \leq 0$ even)

$\mathfrak{X} / \mathcal{O}_C$ as before (in large degrees)

(Like motivic spectral sequence étale ⇒ K-theory)

Fact $TR^\bullet(A; p, \mathbb{Z}_p)$ for A a smooth $\mathcal{O}_C$-algebra can be computed in terms of deRham-Witt groups $W_\bullet\Omega^i_{A/\mathcal{O}_C}$

(Langer and Zink)

Conj For a smooth $\mathcal{O}_C$-algebra A, there is a natural complex $\tilde{\Omega}^\bullet_{A/\mathcal{O}_C}$ of A-modules s.t. $H^i(\tilde{\Omega}^\bullet_{A/\mathcal{O}_C}) = (\Omega^i_{A/\mathcal{O}_C})^\wedge_p$ (*)

(mod Tate twist)

Same for $\widetilde{W_r\Omega^\bullet}$

Moreover, $\tilde{\Omega}^\bullet_{A/\mathcal{O}_C} \otimes_{\mathcal{O}_C} k \cong \Omega^\bullet_{A_k/k}$

Considered as a complex of A_k-modules via F s.t. * becomes the Cartier isomorphism.

Back to reality:

Thm $\widetilde{W_r\Omega^\bullet_{A/\mathcal{O}_C}}$ exist with all expected properties.

Then, $R\Gamma_{A_{\inf}}(\mathfrak{X}) = R\varprojlim_{r,F} R\Gamma(X, \widetilde{W_r\Omega^\bullet})$

relies on 2 observations:

1) If $\nu : X_{\text{proét}} \to \mathfrak{X}$ projection, then

$$(S.)\quad R^i\nu_*\widehat{\mathcal{O}}_X = \Omega^i_{\mathfrak{X}}[\tfrac{1}{p}](-i)$$

Thus $R\nu_*\widehat{\mathcal{O}}_X$ does the job rationally. But, $R^i\nu_*\widehat{\mathcal{O}}_X^+$ has some junk torsion, killed by $\zeta_p - 1$

2) Let $L\eta_{\zeta_p-1}K^\bullet$ for a complex $K^\bullet$ of flat $\mathcal{O}_C$-modules be

$$(L\eta_{\zeta_p-1}K^\bullet)^i = \{x \in (\zeta_p-1)^iK^i \mid dx \in (\zeta_p-1)^{i+1}K^{i+1}\}$$

this kills the junk torsion! Take

$$\widehat{\Omega}^\bullet \underset{(\text{almost})}{=} L\eta_{\zeta_p-1} R\nu_* \widehat{\mathcal{O}_X^+}$$