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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Pierre Colmez

Talk Title: Finite dimensional Banach Spaces (MSRI/Evans Lecture)

Date: 12 / 1 / 2014 **Time:** 4 : 10 am / [✓ pm] (circle one)

List 6-12 key words for the talk: Vector Spaces, periods, p-adic Hodge theory, comparison theorems

Please summarize the lecture in 5 or fewer sentences: Introduces Vector Spaces (Functors from nice algebras to $\mathbb{Q}_p$-vector spaces) in the context of p-adic HT and Fontaine's period rings and gives an example application to comparison theorems.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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MSRI 719 – Automorphic forms, Shimura varieties, Galois representations and L-functions

MSRI/Evans Lecture – Finite dimensional Banach Spaces

Pierre Colmez

Dec 01, 2014 – 4:10pm-5:00pm

Finite dimensional Banach Spaces

Fix p prime, and let $|\cdot|_p$ be the p-adic absolute value on $\mathbb{Q}$ with $|p|_p = p^{-1}$. It satisfies $|xy|_p = |x|_p|y|_p$, and $|x+y|_p \leqslant \max(|x|_p, |y|_p)$.

Let $\mathbb{Q}_p = \hat{\mathbb{Q}}$ be the completion, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p, |x|_p \leqslant 1\}$. Can also construct algebraically by $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$, but we will focus on the analytic viewpoint.

Let $\overline{\mathbb{Q}_p}$ be the algebraic closure of $\mathbb{Q}_p$ ($[\overline{\mathbb{Q}_p} : \mathbb{Q}_p] = \infty$ since, e.g., $x^n - p$ is irreducible in $\mathbb{Q}_p[x]$ for all n).

$|\cdot|_p$ extends uniquely to $\overline{\mathbb{Q}_p}$ and $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ acts via isometries.

Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$. Then $G_{\mathbb{Q}_p} \circlearrowright \mathbb{C}_p$ and $G_{\mathbb{Q}_p} = \mathrm{Aut}_{\mathrm{Cont}}(\mathbb{C}_p)$.

Note: $\mathbb{C}_p \cong \mathbb{C}$ as fields, but to find such an isomorphism we need to invoke the axiom of choice, so we should forget that it exists at all. We should think of $[\mathbb{C}_p : \mathbb{Q}_p]$ like $[\mathbb{C} : \mathbb{Q}]$ since both are uncountable.

Tate (1966): $\mathbb{C}_p$ does not contain $2\pi i$. Hint – $\log e^{2\pi i/p^n} = 0$, where

$$\log(x) = \sum \frac{(-1)^n - 1}{n}(x-1)^n.$$

We have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mu_{p^\infty} \arrow[r] & B(1,1^-) \arrow[r, "\log"] & \mathbb{C}_p \arrow[r] & 0 \end{tikzcd}$$

We have

$$\sigma(e^{2\pi i/p^n}) = e^{\chi(\sigma)2\pi i/p^n}$$

and applying log we get $\sigma(2\pi i) = \chi(\sigma)2\pi i$, which by Tate implies $2\pi i = 0$.

Fontaine (1980) constructed a natural ring $B_{dR}^+ \ni 2\pi i = t$ with an action of $G_{\mathbb{Q}_p}$. The action satisfies $\sigma(t) = \chi(\sigma)t$. There is a map $\theta : B_{dR}^+ \to \mathbb{C}_p$ with kernel generated by t.

Note: $B_{dR}^+ \cong \mathbb{C}_p[[t]]$ as rings, but again need the axiom of choice to find such an isomorphism, so we should not think of it this way.

$B_{dR}^+/t^2B_{dR}^+$ is the completion of $\overline{\mathbb{Q}_p}$ for $|\cdot|_{p,1}$ which is defined as follows: for $x \in \overline{\mathbb{Q}_p}$, we can write $x = Q(\pi)$ where $Q \in \mathbb{Q}_p[\mu_n][X]$ and π is killed by an Eisenstein polynomial P. Then

$$\frac{dx}{dp} = \frac{-Q'(\pi)}{P'(\pi)}$$

and

$$|x|_{p,1} = \max(|x|_p, |\frac{dx}{dp}|)$$

There is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & tB^+_{dR}/t^2 \arrow[r] \arrow[d, equal] & B^+_{dR}/t^2 \arrow[r] & B^+_{dR}/t \arrow[r] \arrow[d, equal] & 0 \\
& \mathbb{C}_p t & & \mathbb{C}_p &
\end{tikzcd}$$

Let $U = \{(x_0, x_1, ..., x_n, ...),\, x_n \in B(1, 1^-),\, x_{n+1}^p = x_n\}$. There is a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "\widetilde{\log}"] \arrow[d, "x \mapsto x_0"'] & B^+_{dR} \arrow[d] \\
B(1, 1^-) \arrow[r, "\log"] & \mathbb{C}_p
\end{tikzcd}$$

From this we obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}_p t \arrow[r] & U \arrow[r] & \mathbb{C}_p \arrow[r] & 0
\end{tikzcd}$$

So $U \cong \mathbb{C}_p \oplus \mathbb{Q}_p$ as $\mathbb{Q}_p$-vector spaces.

$B^+_{cris} \subset B^+_{dr}$, $B^+_{cris} \circlearrowleft \varphi$, the Frobenius. The map $\widetilde{\log}$ factors as $\widetilde{\log} : U \to (B^+_{cris})^{\varphi = p}$ (since $\log x^p = p \log x$). More generally, we have

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}_p t^m \arrow[r] & (B^+_{cris})^{\varphi = p^m} \arrow[r] & B^+_{dR}/t^m \arrow[r] & 0
\end{tikzcd}$$

Problem: $\mathbb{C}_p \cong \mathbb{C}_p \oplus \mathbb{Q}_p$ as $\mathbb{Q}_p$ vector spaces; how to distinguish?

Finite dimensional Vector Spaces (2000, Fontaine-Plut, Fargues, Scholze).

A Banach $\mathbb{Q}_p$-algebra ($||xy|| \leqslant ||x||||y||, ||x + y|| \leqslant \max(||x||, ||y||)$) Λ is *nice* if $||x|| = \max_{s:\Lambda \to \mathbb{C}_p} |s(x)|$ and $x \mapsto x^p$ is surjective. E.g., $\Lambda = \mathbb{C}_p$.

A *Vector Space* is a functor from nice algebras to $\mathbb{Q}_p$-vector spaces. Examples:

- V a finite dimensional $\mathbb{Q}_p$-vector space, $V(\Lambda) = V \,\forall\, \Lambda$, $V(\Lambda_1) \xrightarrow{\text{Id}} V(\Lambda_2)$.
- $\mathbb{V}^d$, $\mathbb{V}^d(\Lambda) = \Lambda^d$.

A Vector Space $\mathbb{W}$ is finite dimensional if it can be presented as

$$\begin{tikzcd}
0 \arrow[r] & V_2 \arrow[dr] & & & \\
0 \arrow[r] & V_1 \arrow[r] & \mathbb{W}' \arrow[r] \arrow[dr] & \mathbb{V}^d \arrow[r] & 0 \\
& & & \mathbb{W} \arrow[r] & 0
\end{tikzcd}$$

Define $\dim\mathbb{W} = d$, $\mathrm{ht}\mathbb{W} = \dim_{\mathbb{Q}_p} V_1 - \dim_{\mathbb{Q}_p} V_2$, and $\mathrm{Dim}\mathbb{W} = (\dim\mathbb{W}, \mathrm{ht}\mathbb{W})$.

Theorem. (1) $\mathrm{Dim}\mathbb{W}$ *is well-defined*

(2) *For* $f : \mathbb{W}_1 \to \mathbb{W}_2$, $\ker f$ *and* $\mathrm{Im} f$ *are finite dimensional Vector Spaces, and* $\mathrm{Dim}\mathbb{W}_1 = \mathrm{Dim}\ker f + \mathrm{Dim}\mathrm{Im} f$.

(3) *If* $\dim\mathbb{W} = 0$ *then* $\mathrm{ht}\mathbb{W} \geqslant 0$.

(4) $\mathbb{W} \subset \mathbb{V}^1$ *implies that* $\mathbb{W}$ *is* $\mathbb{V}^1$ *or finite dimensional over* $\mathbb{Q}_p$, *and in particular* $\mathrm{ht}\mathbb{W} \geqslant 0$.

Example. (1) For $m \geqslant 1$, $\mathbb{B}_m = \mathbb{B}^+_{dR}/t^m \mathbb{B}^+_{dR}$. $\mathrm{Dim}\mathbb{B}_m = (m, 0)$.

(2) For a, b, $U_{a,b} = (\mathbb{B}^+_{cris})^{\varphi^a = p^b}$. Dim$U_{a,b} = (b, a)$. Cf. before, where we had

$$\begin{tikzcd}
0 \arrow[r] & \underset{(0,1)}{\mathbb{Q}_p t^m} \arrow[r] & \underset{U_{1,m}\ (m,1)}{(B^+_{cris})^{\varphi = p^m}} \arrow[r] & \underset{\mathbb{B}_m\ (m,0)}{B^+_{dr}/t^m} \arrow[r] & 0
\end{tikzcd}$$

Comparison theorems and periods ($\int_{S^1} \frac{dz}{z} = 2\pi i$)

Let $X/\mathbb{Q}$ be projective and smooth. There is a pairing

$$H^i_{dR}(X(\mathbb{C})) \times H_i(X(\mathbb{C}), \mathbb{Z}) \to \mathbb{C}$$

given by $(\omega, u) = \int_c \omega$. This induces an isomorphism

$$\mathbb{C} \otimes H^i_B(X(\mathbb{C})), \mathbb{Q}) \cong \mathbb{C} \otimes H^i_{dR}(X)$$

Note we have $\mathbb{Q}_p \otimes H^1_B(X, \mathbb{Q}) = H^1_{\text{ét}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$.

There is a comparison theorem

$$B^+_{dR}[1/t] \otimes H^1_{\text{ét}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \cong B^+_{dR} \otimes H^1_{dR}(X)$$

The isomorphism respects the actions of $G_{\mathbb{Q}_p}$ (induced by the action on étale cohomology, the action on B^+_{dR}, and the trivial action on de Rham cohomology) and the filtrations (induced by the powers of t filtration on B^+_{dR}, the trivial filtration on étale cohomology, and the Hodge filtration on de Rham cohomology).

The same is true for B^+_{cris} if X has good reduction, in which case the isomorphism also respects the Frobenius φ.

Thanks to a lot of work, for $r >> 0$, there is an exact sequence

$$\begin{tikzcd}
\ldots \arrow[r] & \underset{(0,a_i)}{H^i_{\text{ét}}} \arrow[r] & \underset{(b_i,c_i)}{(t^{-r} B^+_{cris} \otimes H^i_{dR})^{\varphi=1}} \arrow[r, "\iota"] & \underset{(b'_i,0)}{(t^{-r} B^+_{dR} \otimes H^i_{dR})/\mathrm{Fil}^0} \arrow[r] & \underset{(0,a_{i+1})}{H^{i+1}_{\text{ét}}} \arrow[r] & \ldots
\end{tikzcd}$$

All of these spaces are $\mathbb{C}_p$ points of finite dimensional Vector Spaces, with the Dimensions listed below each term above. We can use this to prove that the exact sequence splits into short exact sequences, i.e. that ι is surjective: First, observe that because the codomain of ι is a successive extension of $\mathbb{V}^1$'s, the fourth property of Dim in the theorem above implies that ht(Imι) $\geqslant 0$. This implies ht(cokerι) $\leqslant 0$ (their sum is 0). On the other hand, dim(cokerι) $= 0$ because it injects into a space of dimension $(0, a_{i+1})$, and thus ht(cokerι) $\geqslant 0$. Thus Dim(cokerι) $= (0, 0)$ and coker$\iota = 0$.