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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Wei Zhang

Talk Title: Special parahorics and exotic good reduction

Date: 12 / 02 / 2014 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Arithmetic transfer conjecture, Fundamental lemma, special parahoric, moduli of p-divisible group

Please summarize the lecture in 5 or fewer sentences: Begins by describing a moduli space of unitary p-divisible groups with "exotic" good reduction - that is, the level is not hyper-special but the reduction is smooth. Then describes an arithmetic transfer conjecture relating derivatives of orbital integrals on related groups to intersection numbers of certain intersections in the moduli space, and some results towards it.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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Gan Gross Prasad

arithmetic setting (unitary groups)

Gross-Zagier height pairing $\leftrightarrow$ Derivative of L-function

$\hookrightarrow$ Local problems: arithmetic fundamental lemma
(For unitary groups unramified at ∞ place)

<u>Today</u> $F \,|\, F_0 = \mathbb{Q}_p$ quadratic extension $\pi \,|\, \varpi$ $\pi^2 = \varpi$ ramified ($p \neq 2$)

$k = \mathcal{O}_{F_0}/(\varpi)$

I) Special parahorics

V a vector space over F, dim $= n$ Hermitian space $(\ ,\ )$
(split)

$\Lambda \subseteq V$ is $\begin{cases} n \text{ even: } \pi\text{-modular} \Leftrightarrow \Lambda \subseteq \Lambda^\vee = \frac{1}{\pi}\Lambda \\ n \text{ odd: nearly } \pi\text{-modular} \Leftrightarrow \Lambda \subseteq \Lambda^\vee \underset{1}{\subsetneq} \frac{1}{\pi}\Lambda \end{cases}$

$\Lambda^\vee = \{x \in V : (x, \Lambda) \subseteq \mathcal{O}_F\}$

length 1

$K \subseteq U(V)$ is stab Λ

Remark: $\dim \Lambda^\vee/\Lambda = 2\lfloor \frac{n}{2} \rfloor$
and inherits a symplectic, non-degenerate form from $(\ \ )$

II) Formal moduli space.

$\breve{F}$ = completion of F^{ur}

Def'n Unitary p-div. group of sign $(1,n-1)$ over $S \in \mathrm{Nil}_{p, \mathcal{O}_{\breve{F}}}$ p locally nilpotent

$$(X, \iota_X, \lambda_X)$$

$\iota_X : \mathcal{O}_F \hookrightarrow \mathrm{End}(X)$, $\lambda_X : X \to X^\vee$ polarization

s.t.

1) $\mathrm{Ker}(\lambda) = \begin{cases} X[\pi] & n = \text{even} \\ \subseteq X[\pi] \text{ of order } q^{n-1} & n = \text{odd} \end{cases}$

2) Kottwitz sign $(1, n-1)$

3) Wedge condition $\Lambda^2(\iota(\pi) + \pi \mid \mathrm{Lie}\, X) = 0$

4) Spin condition

$n = \text{even}$: $\iota(\pi)|_{\mathrm{Lie}(X)}$ non-vanishing pointwise

harder to state for n odd

5) $\left. \begin{array}{l} S = \mathrm{Spec}\, \bar{k} \\ \text{supersingular} \end{array} \right\} \Rightarrow$ unique unitary p-div. group

Framing object $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$

Remark $\mathrm{Aut}(\mathbb{X}_n) = \{ g \in \mathrm{Aut}_{\mathcal{O}_F}(\mathbb{X}_n) \ \ gg^\dagger = 1 \ \ (g^\dagger \text{ Rosati involution}) \}$

$$\updownarrow$$

$U(V_{\mathbb{X}_n})$ $V_{\mathbb{X}_n}$ = non-split Hermitian

Formal moduli space

$$S \mapsto \{ (X, \iota_X, \lambda_X, \rho_X) \}$$

$$\rho_X : X \times_S \bar{S} \to \mathbb{X}_n \times_{\bar{\mathbb{F}}} \bar{S} \text{ quasi-isogeny of height } 0$$

<u>Theorem</u> $\mathcal{N}_n$ formally smooth essentially proper of relative dimension $n-1$

Note: Not hyperspecial level but still good reduction.

III) Intersection number

$n = \text{odd}$

$$\mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_n$$

$$(X \cdots) \mapsto (X \times \bar{\mathcal{E}}, \cdots)$$

$-$ is conjugation here for signature

($\underline{\text{Ex}}$: $n=1$, $\mathcal{N}_1 = \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$, $\mathcal{E}$ universal object)

$$\Delta : \mathcal{N}_{n-1} \to \mathcal{N}_{n-1} \times \mathcal{N}_n$$

↺ "non-split" unitary group G_1

G_0 = unitary group attached to split space

$G_1 = U(\mathbb{X}_{n-1}) \times U(\mathbb{X}_n)$ $\cdots$ non-split

$g \in G_1$, $\mathrm{Int}(g) = (\Delta, g\Delta) = \chi(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{g\Delta})$

(only for g regular semisimple — $U(\mathbb{X}_{n-1}) \xrightarrow{\text{diagonal}} G_1$, $U(\mathbb{X}_{n-1}) = H_1$

$\mathrm{Int}(g)$ depends on $H_1 \backslash G_1 / H_1$. Regular s.s. $\Leftrightarrow$ trivial stabilizer and orbit is closed in G_1)

IV) Orbital integrals

$G' = GL_{n-1,F} \times GL_{n,F}$

$H_1' = \Delta\, GL_{n-1,F}$ $\quad\quad$ $H_2' = GL_{n-1} F_0 \times GL_n F_0$

$\left(\begin{array}{l} GL_{n-1}(F) \to GL_n(F) \\ g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right)$ $\quad\quad$ $H_i' \subset G'$

$$f' \in C_c^\infty(G'(F_0)) \quad \gamma \in G'(F_0)$$

$$\mathrm{Orb}(\gamma, f'(s)) = \iint f'(h_1^{-1}\gamma h_2)\,\eta(h_2)\,|\det h_1|^s\, dh_1\, dh_2$$

$\eta_{F/F_0} : H_2'(F_0) \to \{\pm 1\}$ quadratic character (ramified) (compose with det.)

Relation to unitary setting

$H_i \leq G_i \quad i = 0, 1$

$$[H_1' \backslash G' / H_2']_{\text{reg. s.s.}} = [H_0 \backslash G_0 / H_0]_{\text{reg. s.s.}} \sqcup [H_1 \backslash G_1 / H_1]_{\text{reg. s.s.}}$$

<u>Def.</u> $f' \in C_c^\infty(G'(F_0))$, (f_0, f_1) on G_i

match if $\omega(\gamma)\,\mathrm{Orb}(\gamma, f', 0) = \begin{cases} \mathrm{orb}(g, f_0) & g \in G_0 \\ \mathrm{orb}(g, f_1) & g \in G_1 \end{cases}$

$\forall\ \gamma \leftrightarrow g$

<u>Fundamental Lemma</u> F/F_0 unramified

$$\mathbb{1}_{G'(\mathcal{O}_{F_0})} \leftrightarrow (\mathbb{1}_{K_0}, 0) \quad (\text{Yun } p \gg 0)$$

V) Arithmetic transfer conj. Zhang (5)

AFL: $\gamma \leftrightarrow g \in G_1$

$$\omega(\gamma) \left.\frac{d}{ds}\right|_{s=0} \mathrm{Orb}(\gamma, \mathbb{1}_{G'(\mathcal{O}_F)}, s) = \mathrm{Int}(g)$$

(F/F_0 unramified)

($n=2,3$ Known)

F/F_0 ramified $K = K_{n-1} \times K_n < G_0$
Parahorics

$$f' \leftrightarrow (\mathbb{1}_K, 0)$$

Question: explicit transfer function f'?

<u>Conj</u> (joint w/ Rapoport, Smithling)

i) For any $f' \leftrightarrow (\mathbb{1}_K, 0)$, we have

$$\left(\left.\frac{d}{ds}\right|_{s=0} \mathrm{Orb}(\gamma, f', s)\right) + \mathrm{Orb}(\gamma, f'_{corr}, 0) = \mathrm{Int}(g)$$

$\gamma \leftrightarrow g$

For some $f'_{corr} \in C_c^\infty(G'(F_0))$

ii) One can choose f' s.t. $f'_{corr} = 0$

Remark: ① Some harmonic analysis $\Rightarrow$ (i) $\Leftrightarrow$ (ii)
② (i) $\Rightarrow$ (ii)

<u>Theorem</u> $n=3$ conj. holds

<u>proof</u> (ingredients)

$\mathrm{Int}(g)$ can be calculated explicitly

$\mathcal{N}_2$ = def. space of 1-dim ht 2 $\longleftrightarrow$ germ expansion.