NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe  Email/Phone: sean.p.v.h@gmail.com
Speaker's Name: Dipendra Prasad
Talk Title: Branching laws and period integrals for non-tempered representations
Date: 12/03/2014  Time: 12:00 am / pm (circle one)
List 6-12 key words for the talk: Branching laws, non-tempered representations,
classical groups
Please summarize the lecture in 5 or fewer sentences: Explains some branching
laws for restriction of non-tempered automorphic representations
for classical groups, with many explicit examples and applications.

CHECK LIST

(This is NOT optional, we will not pay for incomplete forms)

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Branching laws for non-tempered representations

Dipendra Prasad
Tata Institute of Fundamental Research

Automorphic forms, Shimura varieties, Galois representations, and $L$-functions

MSRI

December 03, 2014

(joint work with Wee Teck Gan and B. Gross)
Branching laws for compact unitary groups (from $U(n+1)$ to $U(n)$):

\[ \lambda = \{ \lambda_1 \geq \cdots \geq \lambda_{n+1} \} \]

\[ \pi_\lambda|_{U(n)} = \sum \pi_\mu, \]

where $\mu$ runs over

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Two features of this branching law may be noted.

1. Multiplicity one.
2. Explicit description depends on a parametrization of all irreducible representations, in this case by the theory of highest weights.
We are interested in similar branching laws for real and $p$-adic groups for representations which are typically infinite dimensional. I will concentrate mostly on the $p$-adic case where we will consider representations of a $p$-adic group on a vector space over $\mathbb{C}$, and the representations will be smooth.
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\[ \text{Hom}_H(\pi_1, \pi_2) \neq 0. \]
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Branching laws will be understood in the sense of

$$\text{Hom}_H(\pi_1, \pi_2) \neq 0.$$ 

It may be remarked that a priori the space

$$\text{Hom}_H(\pi_1, \pi_2)$$

may be identically zero, or may be identically infinite dimensional!
Branching laws that we consider are for pairs of groups and subgroups which are:

- $GL_{n+1} \supseteq GL_n$
- $SO_{n+1} \supseteq SO_n$
- $U_{n+1} \supseteq U_n$

and some more which go under the name of Bessel subgroup, and Fourier-Jacobi subgroup, but these we will not discuss here.
One of the first theorems that one proves for all these branching laws is the following multiplicity one theorem.

Theorem (Aizenbud, Gurevitch, Rallis, Schiffmann)

For groups \((G, H)\) as above, 

\[
\dim \text{Hom}_H(\pi_1, \pi_2) \leq 1,
\]

for irreducible admissible representations \(\pi_1\) of \(G\), and \(\pi_2\) of \(H\).

Given this theorem, the main question to understand is when 

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Given this theorem, the main question to understand is when

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Introduction

Theorem

For $\pi_1$ an irreducible admissible generic representation of $GL_{n+1}$, and $\pi_2$ of $GL_n$, $\dim \text{Hom}(\pi_1, \pi_2) = 1$. 

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Branching laws for non-tempered representations
Theorem (Waldspurger, Moeglin-Waldspurger, Beuzart-Plessis)

For pair of groups \((G, H)\) as above,

\[\sum_{\pi_1 \in \Pi_1(G)} \sum_{\pi_2 \in \Pi_2(H)} \dim \text{Hom}(\pi_1, \pi_2) \leq 1,\]

where the pairs \(H' \subseteq G'\) vary over all pure inner forms of a given pair \((G, H)\), and \(\Pi_1(G)\) (resp. \(\Pi_1(H)\)) denotes an \(L\)-packet of representations on \(G\) (resp. \(H\)) which contains a generic representation.
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2. \[
\sum_{H' \subseteq G'} \sum_{\pi_1 \in \Pi_1(G'), \pi_2 \in \Pi_2(H')} \dim \text{Hom}(\pi_1, \pi_2) = 1,
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For example for unitary groups over reals, we have the pairs:

\[ U(n, 0) \leftrightarrow U(n + 1, 0) \]
\[ U(n - 1, 1) \leftrightarrow U(n, 1) \]
\[ \vdots \]
\[ U(0, n) \leftrightarrow U(1, n) \]
For a reductive algebraic group $G$ over a local field, if $\Pi(G)$ denotes the set of isomorphism classes of representations of $G$, and $\Sigma(G)$ denotes the set of equivalence classes of (admissible) parameters for $G$, then there is a surjective map with finite fibers:

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Branching laws for non-tempered representations
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Representations of pure inner forms of $G$ with a given parameter $\varphi$ are in bijective correspondence with $\hat{S}_\varphi$, where $S_\varphi$ denotes the group of connected components of the centralizer of the parameter $\varphi$. 
The component groups in the cases being considered are elementary abelian 2 groups, i.e., $\mathbb{Z}/2^d$, explicitly parametrized by irreducible self-dual summands of the correct parity in the representation

$$\varphi : W'_k \rightarrow ^L G \rightarrow GL_n(\mathbb{C}).$$
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\[
\varphi : W'_k \rightarrow L G \rightarrow GL_n(\mathbb{C}).
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The distinguished member \((\pi_{1,0}, \pi_{2,0})\) with
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corresponds to the character on the component group (which is essentially)

\[ S_{\varphi_1} \times S_{\varphi_2} \to \mathbb{Z}/2 \]

\[ \varphi_{1,i} \to \varepsilon(\varphi_{1,i} \otimes \varphi_2) \]

\[ \varphi_{2,i} \to \varepsilon(\varphi_1 \otimes \varphi_{2,i}). \]
Example 1 (Harder, Langlands, Rapoport):

Let $K$ be a quadratic extension of a number field $k$, $\pi$ a cuspidal automorphic representation of $GL_2(\mathbb{A}_K)$ with trivial central character on $\mathbb{A}_k^\times$. Then,

$$\int_{\mathbb{A}_k^\times \, GL_2(k) \backslash GL_2(\mathbb{A}_k)} f(h) dh \not\equiv 0$$
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\int_{\mathbb{A}_k^\times \ GL_2(k) \backslash GL_2(\mathbb{A}_k)} f(h)dh \neq 0
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if and only if \( L(s, \text{As } \pi) \) has a pole at \( s = 1 \).
Example 2 (Gelbart, PS, Rogawski): \( U(1, 1) \hookrightarrow U(2, 1) \).

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(i) $\pi = \bigotimes_{\nu} \pi_{\nu}$ is locally generic at all the places $\nu$;
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if and only if

(i) $\pi = \bigotimes_\nu \pi_\nu$ is locally generic at all the places $\nu$;
(ii) $L(s, BC(\pi))$ has a pole at $s = 1$. 
The non-tempered representations that we will consider in this lecture are those which arise as local components of automorphic representations, and which are in particular unitary representations. These are parametrized by Arthur by a variant of the Weil-Deligne group:

$$\psi: W'_k \times \text{SL}_2(\mathbb{C}) \to LG$$

where $W'_k = W_k$ or $W_k \times \text{SL}_2(\mathbb{C})$ depending on whether $k$ is Archimedean or not, and where $\psi$ restricted to $W_k$ has bounded image in the dual group.
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Let $\varphi_\psi$ be the composition:

$$W'_k \to W'_k \times SL_2(\mathbb{C}) \to L^1 G,$$

where the mapping from $W'_k$ to $SL_2(\mathbb{C})$ is given by the diagonal map $(\nu^{1/2}, \nu^{-1/2})$. 

Associated to $\psi$, Arthur attaches a finite set $\Pi(\psi)$ of representations of $G(k)$ which contains the set of representations in the $L$-packet associated to $\varphi_\psi$. In this lecture we will consider only those representations of $G(k)$ which belong to the $L$-packet associated to the Langlands parameter $\phi_\psi$ associated to an $A$-parameter $\psi$. 

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Theorem for $GL_n$

**Theorem**

Let $\pi_1$ be an irreducible admissible representation of $GL_{n+1}(k)$ with $A$-parameter, i.e., a representation of $W'_k \times SL_2(\mathbb{C})$, given by

$$\sigma_1 = \sum_{i=0}^d (\sigma_{i+1}, \sigma_i) \otimes \text{Sym}^i(C^2),$$

and $\pi_2$ an irreducible admissible representation of $GL_n(k)$ with $A$-parameter (of dimension $n$) given by

$$\sigma_2 = \sum_{i=0}^d \sigma_{i+1} \otimes \text{Sym}^i(C^2) \oplus \sum_{i=0}^d \sigma_{i-1} \otimes \text{Sym}^{i-1}(C^2),$$

then $\dim \text{Hom}(\pi_1, \pi_2) = 1$ for an arbitrary tempered part in $\sigma_2$.

Conversely, if $\dim \text{Hom}(\pi_1, \pi_2) = 1$, then the parameters of $\pi_1$ and of $\pi_2$ can be expressed in this form.
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and $\pi_2$ an irreducible admissible representation of $GL_n(k)$ with $A$-parameter (of dimension $n$) given by

$$\sigma_2 = \sum_{i=0}^{d} \sigma_{2,i} \otimes \text{Sym}^i(\mathbb{C}^2) \oplus \sum_{i=0}^{d} \sigma_{2,i}^{-1} \otimes \text{Sym}^i(\mathbb{C}^2),$$

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then $\dim \text{Hom}(\pi_1, \pi_2) = 1$ for an arbitrary tempered part in $\sigma_2$. Conversely, if $\dim \text{Hom}(\pi_1, \pi_2) = 1$, then the parameters of $\pi_1$ and of $\pi_2$ can be expressed in this form.
Remark: The theorem roughly says that any non-tempered part of $\pi_1$ corresponding to $\text{Sym}^i(\mathbb{C}^2)$ must have a counterpart either in $\text{Sym}^{i+1}(\mathbb{C}^2)$ or $\text{Sym}^{i-1}(\mathbb{C}^2)$, thus the nontempered part of $\pi_1$ determines the nontempered part of $\pi_2$ with finite ambiguity.
Example 1: Classification of representations of $GL_{n+1}$ which carry trivial invariant form for $GL_n$: 

(a) Since the trivial representation of $GL_{n+1}$ corresponds to $\text{Sym}^n(C_2)$, and the trivial representation of $GL_n$ corresponds to $\text{Sym}^{n-1}(C_2)$, this is certainly an allowed branching by our recipe. 

The others being, 

(b) $\text{Sym}^{n-2}(C_2) \oplus$ tempered of $GL_2$
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Example 2:

\[ \pi_n \otimes \text{Sym}^1(\mathbb{C}^2), \]

a Speh module on \( GL_{2n}(k) \) associated to a cuspidal representation \( \pi_n \) of \( GL_n(k) \). In this case the only option for \( \sigma_2 \) by our recipe is,
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a Speh module on \( GL_{2n}(k) \) associated to a cuspidal representation \( \pi_n \) of \( GL_n(k) \). In this case the only option for \( \sigma_2 \) by our recipe is,

\[ \sigma_2 = \pi_n \oplus \text{arbitrary tempered}, \]

so only generic representations appear in this branching.
A paper of Clozel [IMRN, 2004] based on elaboration of Arthur’s work, and the Burger-Sarnak principle, proves that given a reductive subgroup $H$ of a reductive group $G$, there is a map from unipotent conjugacy classes in the $L$-group of $G$ to the unipotent conjugacy classes in the $L$-group of $H$ which underlies the restriction problem in the unitary case (direct integral and all that!),
A paper of Clozel [IMRN, 2004] based on elaboration of Arthur’s work, and the Burger-Sarnak principle, proves that given a reductive subgroup $H$ of a reductive group $G$, there is a map from unipotent conjugacy classes in the $L$-group of $G$ to the unipotent conjugacy classes in the $L$-group of $H$ which underlies the restriction problem in the unitary case (direct integral and all that!), i.e. the restriction of a representation of $G$, with an $A$-parameter containing a unipotent conjugacy class $u_G$ of $^L G$ contains only those representations of $H$ in the spectral decomposition upon restriction to it which have a particular unipotent conjugacy class $u_H$ of $^L H$. 
Clozel’s theorem has been made precise in some cases by A. Venkatesh [2005]. For example in the restriction problem from $\text{GL}_{n+1}(k)$ to $\text{GL}_n(k)$, if the unipotent element in $\text{GL}_{n+1}(\mathbb{C})$ corresponds to the partition $u = n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, then the only unipotent element of $\text{GL}_n(\mathbb{C})$ involved is the one $u' = n_1' \geq n_2' \geq \cdots \geq n_r' \geq 0$, omitting those $n_i$ which are 1, and adding a few 1’s at the end if necessary. There is an analogous statement for induction of unitary representations of $\text{GL}_n(k)$ to $\text{GL}_{n+1}(k)$. 
Clozel’s theorem has been made precise in some cases by A. Venkatesh [2005]. For example in the restriction problem from $GL_{n+1}(k)$ to $GL_n(k)$, if the unipotent element in $GL_{n+1}(\mathbb{C})$ corresponds to the partition $u = n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, then the only unipotent element of $GL_n(\mathbb{C})$ involved is the one $u^- = n_1 - 1 \geq n_2 - 1 \geq \cdots \geq n_r - 1 \geq 0$, omitting those $n_i$ which are 1, and adding a few 1’s at the end if necessary.
Comparison with the work of Clozel and Venkatesh

Clozel’s theorem has been made precise in some cases by A. Venkatesh [2005]. For example in the restriction problem from $\text{GL}_{n+1}(k)$ to $\text{GL}_n(k)$, if the unipotent element in $\text{GL}_{n+1}(\mathbb{C})$ corresponds to the partition $u = n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, then the only unipotent element of $\text{GL}_n(\mathbb{C})$ involved is the one $u^- = n_1 - 1 \geq n_2 - 1 \geq \cdots \geq n_r - 1 \geq 0$, omitting those $n_i$ which are 1, and adding a few 1’s at the end if necessary.

There is an analogous statement for induction of unitary representations of $\text{GL}_n(k)$ to $\text{GL}_{n+1}(k)$. 
The important point to note is that for both induction and restriction questions in this unitary context, one goes from less tempered to more tempered representations (such as in the Harish-Chandra’s Plancherel decomposition for the space $L^2([G \times G]/\Delta(G))$, and in particular, there is no Frobenius reciprocity for unitary representations, whereas we are concerned with admissible representations here which do have Frobenius reciprocity.
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One way to fix this asymmetry, and the corresponding lack of Frobenius reciprocity, is to have the unipotent conjugacy classes $u_1, u_2$ satisfy,

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Our theorem satisfies these in-qualities.
Now we discuss branching laws for classical groups emphasizing the case of orthogonal groups. Thus we discuss the branching laws from $SO(n + 1)$ to $SO(n)$, more generally from $SO(m)$ to $SO(n)$ with $n + 1 \equiv m \text{ mod } 2$ corresponding to Bessel models.
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Let $\psi_1 : W'_k \times \text{SL}_2(\mathbb{C}) \to LSO_m$ and $\psi_2 : W'_k \times \text{SL}_2(\mathbb{C}) \to LSO_n$ be $A$-parameters with the corresponding Langlands parameters $\phi_{\psi_1} : W'_k \to LSO_m$, and $\phi_{\psi_2} : W'_k \to LSO_n$. 
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Let $\pi_1$ be an irreducible admissible representation of say $SO_m(k)$ and $\pi_2$ of $SO_n(k)$ with $m \geq n$ belonging to the $L$-packets associated to the Langlands parameters $\phi_{\psi_1} : W'_k \to LSO_m(\mathbb{C})$, and $\phi_{\psi_2} : W'_k \to LSO_m(\mathbb{C})$. 
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The $L$-groups of the groups $\operatorname{SO}_m(k)$ and $\operatorname{SO}_n(k)$ are the usual orthogonal and symplectic groups which come equipped with their natural representations. When we talk of $L(s, \pi_1 \times \pi_2)$ below, it is for the tensor product of the natural representations of the two $L$-groups involved. We will also need the adjoint representation of the $L$-group which is used to define the adjoint $L$-function.
Conjecture

Let $\pi_1, \pi_2$ be irreducible admissible representations of $\text{SO}_m(k), \text{SO}_n(k)$ belonging to $L$-packets associated to $\phi_{\psi_1}$ and $\phi_{\psi_2}$, with $m > n$, and $m - n \equiv 1 \mod 2$. Then if $\pi_2$ appears in the Bessel model of $\pi_1,$

1. The Langlands parameters $\phi_{\psi_1}$ and $\phi_{\psi_2}$ considered as representations of $W'_k$ inside $\text{GL}_{m'}(\mathbb{C})$ and $\text{GL}_{n'}(\mathbb{C})$ are as in the theorem on $\text{GL}_n(k)$ (the tempered part being arbitrary but of appropriate size).
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2. If the Langlands parameters $\phi_{\psi_1}$ and $\phi_{\psi_2}$ are as in 1., then the (Vogan) $L$-packet of representations has a unique member with $\text{Hom}[\pi_1, \pi_2] \neq 0$. 

Classical groups, the local case

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<th>Conjecture</th>
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2. If the Langlands parameters $\phi_{\psi_1}$ and $\phi_{\psi_2}$ are as in 1., then the (Vogan) $L$-packet of representations has a unique member with $\text{Hom} [\pi_1, \pi_2] \neq 0$.

3. The $\epsilon$-factors constructed out of possible symplectic root numbers just as in the earlier works tells which member of the $L$-packet has the invariant form.

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Branching laws for non-tempered representations
Remarks: 1. For representations $\pi_1$ and $\pi_2$ appearing in the previous conjecture, the $L$-function

$$\frac{L(s + 1/2, \pi_1 \times \pi_2)}{L(s + 1, \text{Ad} \pi_1)L(s + 1, \text{Ad} \pi_2)}'$$

is not zero (but can have a pole) at $s = 0$.

2. For representations $\pi_1$ and $\pi_2$ appearing in the previous conjecture for which the $A$-parameter is discrete, the $L$-function

$$\frac{L(s + 1/2, \pi_1 \times \pi_2)}{L(s + 1, \text{Ad} \pi_1)L(s + 1, \text{Ad} \pi_2)}'$$

has neither a zero nor a pole at $s = 0$. 

In a series of paper by Ginzburg, Jiang, Rallis, and Soudry, the authors construct backward lifting from $GL_n(k)$ to classical groups typically by constructing a representation of a classical group by parabolic induction from the representation of $GL_n(k)$ which sits as a Levi subgroup, taking its Langlands quotient, and then taking some Bessel or Fourier-Jacobi model (which we will still not define!).
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The backward lift from $GL_{2n}(k)$ to $SO_{2n+1}(k)$ can be constructed as follows.
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The backward lift from $GL_{2n}(k)$ to $SO_{2n+1}(k)$ can be constructed as follows. Suppose $\pi$ is a supercuspidal representation of $GL_{2n}(k)$ with symplectic Langlands parameter. One induces (a twist of) $\pi$ from $GL_{2n}(k)$ which is a Levi subgroup of $SO_{4n}(k)$ to $SO_{4n}(k)$, and takes an appropriate Langlands quotient at a point of reducibility, and then compute a Bessel model down to $SO_{2n+1}(k)$. 
An example from the work of Ginzburg, Jiang, Rallis, Soudry

The Langlands parameter of the representation of $SO_{4n}(k)$ which is a Langlands quotient at a point of reducibility of the principal series representation of $SO_{4n}(k)$ is,

$$\sigma \otimes \text{Sym}^1(\mathbb{C}^2) = \sigma(\nu^{-1/2} \oplus \nu^{1/2}).$$
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In this case, $\pi_2$ which is a representation of an odd orthogonal group must have the parameter $\sigma$, and so cannot live on a smaller orthogonal group than $SO_{2n+1}(k)$, and on $SO_{2n+1}$ too, there is no option but to be the backward lift of $\pi_1$. 
Here is the conjecture on period integral of Automorphic representations.
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**Conjecture**

Let $F$ be a number field, and $\Pi_1 \times \Pi_2$ an irreducible automorphic representation of $G = \text{SO}_{n+1}(\mathbb{A}_F) \times \text{SO}_n(\mathbb{A}_F)$ lying in the discrete spectrum, with $H = \text{SO}_n(F)$ a subgroup of $\text{SO}_{n+1}(F)$ defined by a codimension one subspace $W$ of a quadratic space $V$ over $F$. 
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$$\int_{H(F)\backslash H(\mathbb{A}_F)} fdh,$$

is nonzero for some $f$ an automorphic function on $G(\mathbb{A}_F)$ belonging to $\Pi_1 \times \Pi_2$ if and only if:
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1. The Langlands parameters associated to $\Pi_1$ and $\Pi_2$ are in the relationship as in the local theorem on $\text{GL}_n$. 
Conjecture

2. $\text{Hom}_{H(F_v)}[\Pi_{1,v} \otimes \Pi_{2,v}, \mathbb{C}] \neq 0$ for all places $v$ of $F$. 
Classical groups, the global case

Conjecture

2. $\text{Hom}_{H(F_v)}[\Pi_{1,v} \otimes \Pi_{2,v}, \mathbb{C}] \neq 0$ for all places $v$ of $F$.

3. 

$$
\frac{L(s + 1/2, \Pi_{1} \otimes \Pi_{2})}{L(s + 1, \text{Ad} \Pi_{1})L(s + 1, \text{Ad} \Pi_{2})} \\
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Further, if the \( L \)-function condition is satisfied, there is a globally relevant pure inner form \( G' \) of \( G \) with an automorphic representation \( \Pi'_1 \otimes \Pi'_2 \) nearly equivalent to \( \Pi_1 \otimes \Pi_2 \) which is globally distinguished by \( H' \).
Theorem

Let $F$ be a number field, and $\Pi_1 \times \Pi_2$ an irreducible automorphic representation of $G = \text{SO}_{n+1}(\mathbb{A}_F) \times \text{SO}_n(\mathbb{A}_F)$ lying in the discrete spectrum, with $H = \text{SO}_n(F)$ a subgroup of $\text{SO}_{n+1}(F)$ defined by a codimension one subspace $W$ of a quadratic space $V$ over $F$. Then if the Langlands parameters associated to $\Pi_1$ and $\Pi_2$ are in the relationship as in the local theorem on $\text{GL}_n$, then, $L(s + \frac{1}{2}, \Pi_1 \otimes \Pi_2) = L(s + 1, \text{Ad} \Pi_1) L(s + 1, \text{Ad} \Pi_2)$, does not have a pole at $s = 0$, and its zeros at $s = 0$ correspond to zeros of $L(1/2, \Pi)$ where $\Pi$ is a symplectic representation constructed as a tensor product of a subrepresentation of $\Pi_1$ with a subrepresentation of $\Pi_2$ (self-dual of appropriate parity).
A theorem on $L$-functions

**Theorem**

Let $F$ be a number field, and $\Pi_1 \times \Pi_2$ an irreducible automorphic representation of $G = \text{SO}_{n+1}(\mathbb{A}_F) \times \text{SO}_n(\mathbb{A}_F)$ lying in the discrete spectrum, with $H = \text{SO}_n(F)$ a subgroup of $\text{SO}_{n+1}(F)$ defined by a codimension one subspace $W$ of a quadratic space $V$ over $F$. Then if the Langlands parameters associated to $\Pi_1$ and $\Pi_2$ are in the relationship as in the local theorem on $\text{GL}_n$, then,
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which seems to play a large role in these branching laws came up in the work of Ichino and Ikeda who proposed that its non-vanishing should control nonvanishing of period integrals.
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2. The initial suggestion to use epsilon factors in these branching laws is due to Michael Harris. Thanks Michael!
Thank you!