

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe Email/Phone: seanpkh@gmail.com

Speaker's Name: Laurent Fargues

Talk Title: G-bundles and the local Langlands program

Date: 12/05/2014 Time: 9:00 am/pm (circle one)

List 6-12 key words for the talk: Diamonds, Fargues-Fontaine curve, Moduli stack of G-bundles, perfectoid spaces

Please summarize the lecture in 5 or fewer sentences: Introduces a relative version of the Fargues-Fontaine curve and explains that $S \rightarrow G\text{-Bundles on } X_S$ is a stack. Formulates conjectures about perverse sheaves on Bun_G in the spirit of the Langlands program. Formulates many things using the language of diamonds.

CHECK LIST

(This is **NOT** optional, we will **not** pay for **incomplete** forms)

- Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
 - **Computer Presentations:** Obtain a copy of their presentation
 - **Overhead:** Obtain a copy or use the originals and scan them
 - **Blackboard:** Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
 - **Handouts:** Obtain copies of and scan all handouts
- For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list.
(YYYY.MM.DD.TIME.SpeakerLastName)
- Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Background on the curve

Fargues (1)

F perfectoid field of char. $p \rightsquigarrow$ "Curve"
Fontaine $\leftarrow F$

3 incarnations

Adic	Schematic	Diamond
$X_F^{\text{ad}} = Y/e\mathbb{Z}$ $V = \text{Spa}(W(\mathbb{C}_F)) \setminus V_{(p, \mathbb{C}_F)}$ Punctured disk variable = p Coeff $\in F$	$X = \text{Proj}(\bigoplus_{i \geq 0} H^0(X^{\text{ad}}, \mathcal{O}(i)))$ $\mathcal{O}(i) = V_{X/e\mathbb{Z}} A^1 \otimes_{\mathbb{C}_F} \sigma^i$ $Y/e\mathbb{Z}$ Noetherian regular of dim 1	$X^\diamond = \text{Spa}(F) \times \text{Spa}(\mathbb{C}_p)^\diamond / e\mathbb{Z}$ $e = \text{Frob}_F \times \text{Id}$ $= \text{Id}_F \times \text{Frob}_{\mathbb{C}_p}^{-1}$

F alg. closed: $k_F \hookrightarrow \mathcal{O}_F \twoheadrightarrow k_F$ Y_F
 $L = W(k_F) \otimes_{\mathbb{C}_F} \mathbb{C}$ \downarrow
 $\text{Spa}(L)$

Isocrystal $\rightarrow \text{Bun}_{X^{\text{ad}}} \xrightarrow{\text{GAGA}} \text{Bun}_X$
 $e\text{-Mod}_L$
 $(D, \mathcal{G}) \cong 1 \rightarrow Y_{X/e\mathbb{Z}} \rightarrow D \rightarrow Y/e\mathbb{Z}$

Theorem (Fontaine-F): It is essentially surjective Fargues ②

Classification of G-Bundles

G a reductive group / k_p

$$G\text{-bond} / X = \begin{cases} G\text{-torsor on } X \\ \text{Bundle} + G \text{ structure (transition)} \end{cases}$$

$$b \in G(L) \mapsto \Sigma_L = G\text{-torsor} / X \rightarrow \text{Bun } X$$

$$\text{Rep}_G \rightarrow \text{e-Mod}_L$$

$$(V, \rho) \mapsto (V_L, \rho(b)\sigma)$$

preceding functor

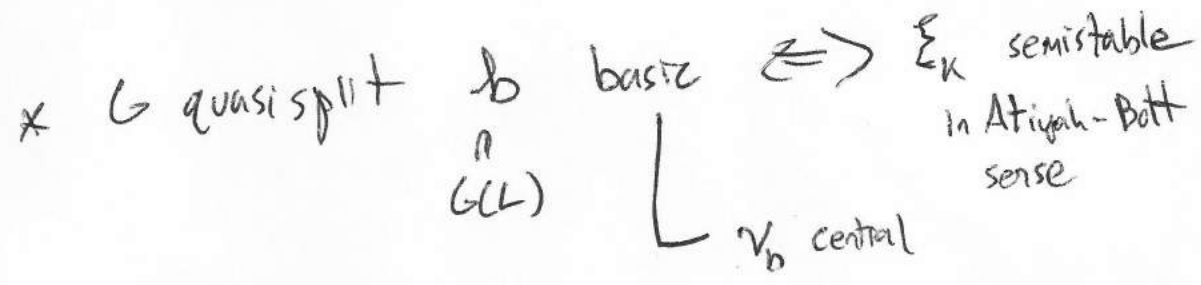
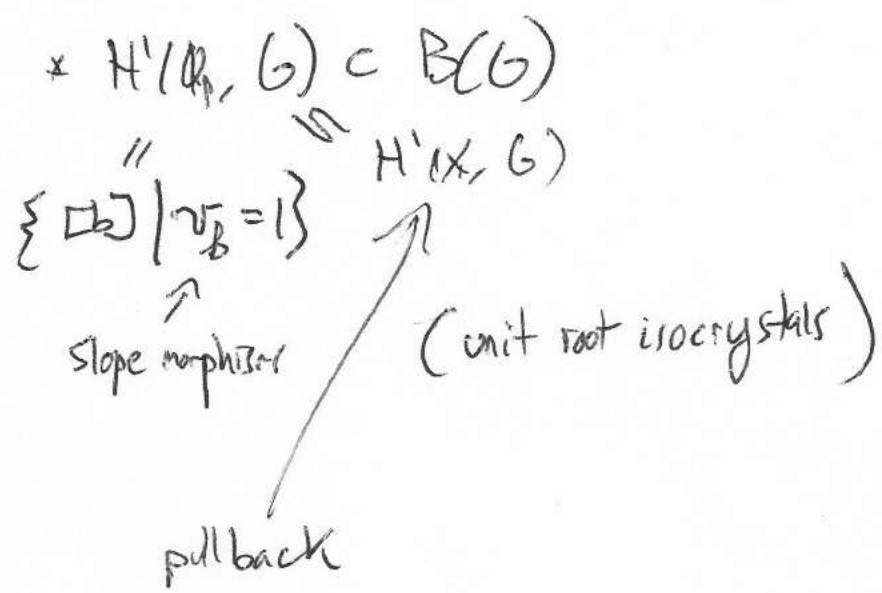
$$\Sigma_b$$

Th * Any G -bundle on X is isomorphic to Σ_b for some $b \in G(L)$

* Canon. bijection $H_{\text{ét}}^1(X, G) \xrightarrow{\sim} B(G)$
 $G(L) / G\text{-conj.}$

(Doesn't depend on choice of section of residue field)

Nice features



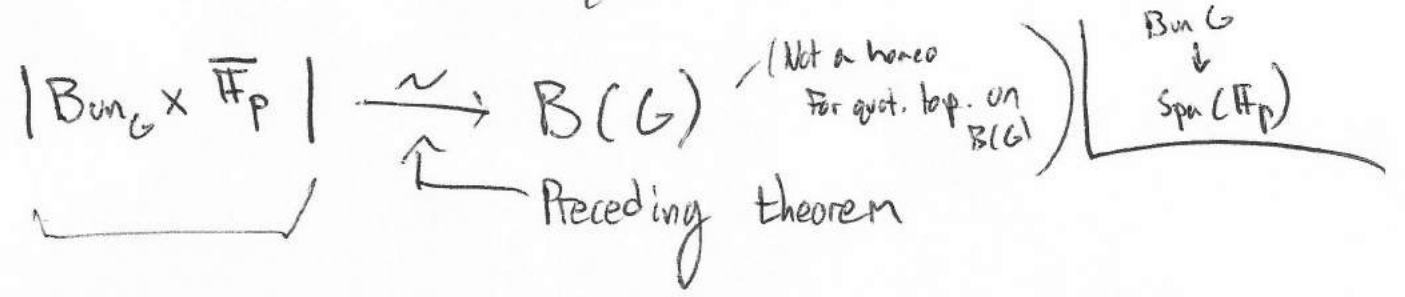
$v_{\mathfrak{b}} = H, N$ -Polygon

Bun G : Perf = \mathbb{F}_p -perfectoid spaces in the sense of Scholze + pro-étale topology

$S \rightarrow X_S^{ad}$ = adic space / exp
 = family of the preceding parameterized by S

Th $S \rightarrow \{ G\text{-bundles on } X_S \}$ is a stack (Scholze, for the faithful top.)

$$X_S^\diamond = (S \times \text{Spa}(\mathbb{Q}_p^\diamond))_{\varphi, \mathbb{Z}}$$



Top space, open = $|U|$ where $U \subseteq \text{Bun}_G / \overline{\mathbb{F}_p}$ open substack

Connected components:

$$\chi: B(G) \rightarrow \pi_1(G) \xleftarrow{\sim} \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{Q}_p)$$

$$\parallel$$

$$X^*(Z(\hat{G})^\Gamma)$$

Ex $G = GL_n, \chi: |\text{Bun}_{\overline{\mathbb{F}_p}}| \rightarrow \mathbb{Z}$
 is the degree of a vector bundle

Th (In progress) χ is locally constant
Scholze F

Conjecture Fibers of χ = connected components

H. N. stratification $| \text{Bun}_G \times \overline{\mathbb{F}_p} | \rightarrow \text{Hom} \left(\text{ID}_{\overline{\mathbb{F}_p}}^+, \frac{G}{\overline{\mathbb{F}_p}} \right)$
slope pos-tors \parallel $X_*(T)_{\mathbb{Q}}$

$$B(G) \supset [b] \longrightarrow [v_b]$$

Kedlaya-Liu This is semi-continuous.
semi-stable locus = open.

S.S. locus : $[b] \in B(G)$

$$\rightsquigarrow \Sigma_b = G\text{-torsor} / X_{\mathbb{F}_p}$$

$G_m \times G_n$ ex. that SS as G -bundle
 $(\mathcal{O} \oplus \mathcal{O}(1))$ \neq SS as v.b.
semi-stable in A-B sense

$$\tilde{J}_b = \text{Aut}(\Sigma_b) = \text{Group shaf on Perf}_{\mathbb{F}_p}$$

diamond // $\pi_0(\tilde{J}_b) = J_b(\mathbb{Q}_p) = \sigma\text{-centralizer of } b(G)$

$$\tilde{J}_b^0 = \text{unipotent}$$

$$\text{Aut}(\mathcal{O} \oplus \mathcal{O}(1)) = \begin{pmatrix} \mathbb{Q}_p^\times & (\mathbb{Z}/p\mathbb{Z}) \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$$

cryptic remark due to scholze $\rightarrow D(\mathbb{Q}_p)$

b basic $\tilde{J}_b^0 = \{1\}$ $\tilde{J}_b = J_b(\mathbb{Q}_p)$

Kedlaya-Liu

"Th" b basic ^{s.s.}

6

$$\text{Bun}_{G/\mathbb{F}_p}^{\text{cbj}} = \mathcal{B}(\mathcal{J}_b(\mathbb{Q}_p))$$

↑ classifying stack.

Motivation

* F perfectoid field,

$$\text{Pic}^0(X_F) = \text{Hom}(\pi_1^{\text{ét}}(X_F), \mathbb{Q}_p^\times)$$

* Kottwitz conjectures (gen. of Vogan)

local Langlands for extended pure inner forms of G

$\mathcal{J}_b, b \text{ basic}$

Kalethe's work for depth zero

$$* \mathbb{Q}_p^\diamond / \varpi^z = X_{\mathbb{F}_1} \rightsquigarrow X_S = S \times S_p(\mathbb{Q}_p)^\diamond / e^z$$

$$* \pi_1 = \text{Gal}(\overline{\mathbb{F}_p} / \mathbb{F}_p)$$

Frankel-Gaitsgory-Vilonen: $\hat{G} = \overline{\mathbb{Q}_p}$ -Langlands dual group

$$\Gamma = \text{Gal}(\overline{\mathbb{F}_p} / \mathbb{F}_p)$$

\hat{G} -torsors $(\mathbb{Q}_p^\diamond / e^z)_{\text{pro-ét}} \rightarrow$ Perverse sheaves on $\text{Bun}_{G/\mathbb{F}_p}$

Conjecture G quasi-split $(B, \psi) =$ Whittaker datum

$$\mathcal{L}_G = [\text{Hom}(W_{\mathbb{F}_p}, {}^L G) / \widehat{G}\text{-conj.}]$$

$\mathcal{L}_G^{\text{disc.}}$ = discrete \mathcal{L} param

$$e = W_{\mathbb{F}_p} \rightarrow {}^L G$$

$$S_e = \text{Cent}_{\widehat{G}}(e) \simeq Z(\widehat{G})^\Gamma$$

$S_e / Z(\widehat{G})^\Gamma$ is finite

$$\exists \text{ functor } \mathcal{L}_G \longrightarrow \text{Perv}(\text{Bun}_G \times \overline{\mathbb{F}_p})^{\text{Weil}}$$

$$\begin{array}{ccc} e & \longmapsto & F_e \\ \downarrow & & \downarrow \\ S_e & \xlongequal{\quad} & S_e \end{array}$$

s.t. 1) If $\xi \in X^*(Z(\widehat{G})^\Gamma) = \pi_0(\text{Bun}_G / \overline{\mathbb{F}_p})$

and $\text{Bun}_G^{\mathbb{Z}} \times \overline{\mathbb{F}_p} = X^{-1}(\xi)$

$F_e |_{\text{Bun}_G^{\mathbb{Z}} / \overline{\mathbb{F}_p}} \simeq Z(\widehat{G})$ acts via ξ

2) $F_e |_{\text{Bun}_G}^{\text{s.s.}} = \text{local system}$
 $j: \text{Bun}_{\text{open}}^{\mathbb{F}_p} \hookrightarrow \text{Bun}$

$$F_e = \text{check } j_*(j^* F_e)$$

e elliptic (e(I_{op}) finite)
 $F_e = 0!$ (j^{*}F)

3) $[b] \in B(G)_{\text{base}}$

$$X_b : \text{Spa}(C_p^b) \xrightarrow{\Sigma_b} \text{Bun } \overline{\mathbb{F}_p}$$

(8)

$$X_b^* \mathcal{F}_\varphi = \bigoplus_{\rho \in \widehat{S_e}} \left[\mathcal{F}_{e,p} \otimes \rho \right]$$

\downarrow
 $S_e \times J_b(\mathbb{Q}_p) \quad \rho \in \widehat{S_e} \quad \rho \in \widehat{S_e} \quad \rho \in \widehat{S_e}$

L-packet

$$\mathcal{L} \mapsto \left(\mathcal{F}_{e,p} \right)_\rho$$

4) Character sheaf -----

5) Hecke property :

$$\text{Bun } G \xleftarrow{c_1 \text{ Hecke}_\nu} \text{Bun } G \times \text{Spa}(\mathbb{Q}_p) \xrightarrow{c_2}$$

$$c_2! (c_1^* \mathcal{F}_e \otimes \chi_\nu) = \mathcal{F}_e \boxtimes \Omega_\nu \circ \mathcal{L}$$

A few things: * OK for tori. (\Leftrightarrow local class field theory)

* Specialize Hecke at the trivial vector bundle (X_1^*)

Find Kottwitz conj. for $\text{Het}(\mathbb{RZ} \text{ spaces})$

* $G = \text{GL}_n, \nu = (1, 0, \dots, 0)$

Scholze conjecture in February about Perrine sheaves pushed forward by Π_{HT}

$$G = \text{GL}_2$$

$$b = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}$$

$$\mathcal{E}_b = \mathcal{O} \oplus \mathcal{O}(1)$$

$$\text{Aut} = \begin{pmatrix} \mathbb{Q}_p^* & \text{Basis} \\ 0 & \mathbb{Q}_p^* \end{pmatrix}$$

$$C_b^* \mathcal{F}_b$$

If $l \neq p$, act trivially.