MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

# NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe Email/Phone: seanpkh@gmail.com

Speaker's Name: Laurent Fargues

Talk Title: G-bundles and the local Langlands program

Date: 12/05/2014 Time: 9:00 am / pm (circle one)

List 6-12 key words for the talk: Diamonds, Fargues-Fontaine curve, Moduli stack of G-bundles, perfectoid spaces

Please summarize the lecture in 5 or fewer sentences: Introduces a relative version of the Fargues-Fontaine curve $X_S$ and explains that $S \to$ G-Bundles on $X_S$ is a stack. Formulates conjectures about perverse sheaves on $Bun_G$ in the spirit of the Langlands program. Formulates many things using the language of diamonds.

# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

# Background on the curve

F perfectoid field of char. p $\rightsquigarrow$ "Curve"
Fontaine-F

3 incarnations

| Adic | Schematical | Diamond |
|---|---|---|
| $X_F^{ad} = Y/\varphi^{\mathbb{Z}}$ | $X = \mathrm{Proj}\left(\bigoplus_{i\geq 0} H^0(X^{ad}, \mathcal{O}(d))\right)$ | $X^\diamond = \mathrm{Spa}(F) \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond / \varphi^{\mathbb{Z}}$ |
| $Y = \mathrm{Spa}(W(\mathcal{O}_F)) \setminus V(p, [\varpi])$ | $\mathcal{O}(d) = Y \times_{\varphi^{\mathbb{Z}}} \mathbb{A}^1$ $\circlearrowleft_{\times p^{-d}}$ | $\varphi = \mathrm{Frob}_F \times \mathrm{Id}$ |
| | | $= \mathrm{Id} \times \mathrm{Frob}^{-1}_{\mathbb{Q}_p^\diamond}$ |
| Punctured disc<br>variable $= p$<br>Coeff $\in F$ | $Y/\varphi^{\mathbb{Z}}$ | |
| | Noetherian regular of dim 1 | |

F alg. closed: $k_F \hookrightarrow \mathcal{O}_F \twoheadrightarrow k_F$

$L = W(k_F)_{\mathbb{Q}} \circlearrowleft \sigma$

$$Y_F \to \mathrm{Spa}(L)$$

$$\underbrace{\text{Isocrystal}}_{\varphi\text{-Mod}_L} \longrightarrow \mathrm{Bun}_{X^{ad}} \xrightarrow{\text{GAGA}} \mathrm{Bun}_X$$

$$(D, \varphi) \longmapsto Y \times_{\varphi^{\mathbb{Z}}} D \to Y/\varphi^{\mathbb{Z}}$$

Theorem (Fontaine-F): It is essentially surjective Fargues ②

---

Classification of G-Bundles

$G$ a reductive group / $\mathbb{Q}_p$

$$G\text{-bund}/X = \begin{cases} G\text{-torsor on } X \\ \text{Bundle} + G \text{ structure (Tannakian)} \end{cases}$$

$b \in G(L) \longmapsto \mathcal{E}_L = G\text{-torsor}/X \longrightarrow \text{Bun}\,X$

preceding functor

$\text{Rep}_{\mathbb{Q}_p} G \longrightarrow \varphi\text{-Mod}_L$

$(V, \rho) \longmapsto (V_L, \rho(b)\sigma)$

$\mathcal{E}_b$

<u>Th</u> * Any $G$-bundle on $X$ is isomorphic to $\mathcal{E}_b$ for some $b \in G(L)$

* <u>Canon.</u> bijection $H^1_{\text{ét}}(X, G) \xrightarrow{\sim} B(G)$

$B(G) = G(L)/\sigma\text{-conj.}$

(Doesn't depend on choice of section of residue field)

Nice features

* $H^1(\mathbb{Q}_p, G) \subset B(G)$

$\{[b] \mid \nu_b = 1\} = H^1(\mathbb{Q}_p, G) \cong H^1(X, G)$

slope morphism

(unit root isocrystals)

pullback

* $G$ quasisplit $b \in G(L)$ basic $\iff \mathcal{E}_b$ semistable in Atiyah-Bott sense

$\nu_b$ central

$\nu_b$ = H.N. Polygon

Bun $G$ : Perf = $\mathbb{F}_p$-perfectoid spaces in the sense of Scholze + pro-étale topology

$S \mapsto X_S^{ad}$ = adic space / $\mathbb{Q}_p$

= family of the preceding parameterized by $S$

$s \in S \quad X_{k(s)} \to X_S$

Th $S \mapsto \{G\text{-bundles on } X_S\}$ is a stack (Scholze, for the faithful top.)

$$X_S^\diamond = (S \times \mathrm{Spa}\,\mathbb{Q}_p^\diamond)_{\varphi^{\mathbb{Z}}}$$

$$|\mathrm{Bun}_G \times \overline{\mathbb{F}}_p| \xrightarrow{\sim} B(G)$$

(Not a homeo For quot. top. on $B(G)$)

$\mathrm{Bun}_G \to \mathrm{Spa}(\overline{\mathbb{F}}_p)$

Preceding theorem

Top space, open $= |\mathcal{U}|$ where $\mathcal{U} \subseteq \mathrm{Bun}_G / \overline{\mathbb{F}}_p$ open substack

Connected components:

$$\kappa : B(G) \to \pi_1(G)_\Gamma \leftarrow \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$

$$\pi_1(G)_\Gamma = X^*(Z(\hat{G})^\Gamma)$$

<u>Ex</u> $G = GL_n$, $\kappa : |\mathrm{Bun}_{\overline{\mathbb{F}}_p}| \to \mathbb{Z}$ is the degree of a vector bundle

<u>Th</u> (In progress Scholze F) $\kappa$ is locally constant

<u>Conjecture</u> Fibers of $\kappa$ = connected components

<u>H. N. stratification</u> $|\mathrm{Bun}_G \times \overline{\mathbb{F}}_p| \to \mathrm{Hom}(\mathbb{D}_{\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p})$

slope pro-torus

$$\mathrm{Hom}(\mathbb{D}_{\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p}) = X_*(T)^+_{\mathbb{Q}}$$

$$B(G) \ni [b] \longmapsto [V_b]$$

Kedlaya - Liu  This is semi-continuous.

semi-stable locus = open.

S.s. locus : $[b] \in B(G)$

$$\rightsquigarrow \mathcal{E}_b = G\text{-torsor}/X_{\mathbb{C}_p^b}$$

$G_m \times G_m$ ex that ss as G-bundle

$\mathcal{O} \oplus \mathcal{O}(1)$ $\neq$ ss as v.b.

semistable in A-B sense

$$\tilde{J}_b = \mathrm{Aut}(\mathcal{E}_b) = \text{Group sheaf on } \mathrm{Perf}_{\mathbb{C}_p^b}$$

//

diamond

$$\pi_0(\tilde{J}_b) = J_b(\mathbb{Q}_p) = \sigma\text{-centralizer of } b(G)$$

$$\tilde{J}_b^0 = \text{unipotent}$$

$$\mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}(1)) = \begin{pmatrix} \mathbb{Q}_p^\times & (B_{cris}^+)^{\varphi = p} \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$$

Cryptic remark due to Scholze

$$\mathcal{D}(\mathbb{Q}_p)$$

b basic  $\tilde{J}_b^0 = \{1\}$  $\tilde{J}_b = J_b(\mathbb{Q}_p)$

Kedlaya-Liu "Th" $b$ basic $\nearrow$ s.s.

$$\mathrm{Bun}_{G/\mathbb{F}_p}^{[b]} = B(J_b(\mathbb{Q}_p))$$

classifying stack.

Motivation * $F$ perfectoid field,

$$\mathrm{Pic}^0(X_F) = \mathrm{Hom}(\pi_1^{geo}(X_F), \mathbb{Q}_p^\times)$$

* Kottwitz conjectures (gen. of Vogan)

local Langlands for extended pure inner forms of $G$ $= J_b$, $b$ basic

Kalethe's work for depth zero

$$* \ \mathbb{Q}_p^\diamond/\varphi^{\mathbb{Z}} = X_{\mathbb{F}_1} \text{ `` }\rightsquigarrow\text{ '' } X_S = S \times \mathrm{Spd}(\mathbb{Q}_p)^\diamond/\varphi^{\mathbb{Z}}$$

"$\pi_1 = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$"

Frenkel-Gaitsgory-Vilonen: $\hat{G} = \overline{\mathbb{Q}}_\ell$-Langlands dual group $\circlearrowleft$ $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

$$\hat{G}\text{-torsors } (\mathbb{Q}_p^\diamond/\varphi^{\mathbb{Z}})_{\text{pro-ét}} \longrightarrow \text{Perverse sheaves on } \mathrm{Bun}_G$$

Conjecture $G$ quasi-split $(B, \psi)$ = Whittaker datum

$$\mathcal{L}_G = \left[ \mathrm{Hom}(W_{\mathbb{Q}_p}, {}^L G) / \hat{G}\text{-conj.} \right]$$

$\mathcal{L}_G^{disc.}$ = discrete $\mathcal{L}$ param

$e = W_{\mathbb{Q}_p} \to {}^L G$

$S_e = \mathrm{Cent}_{\hat{G}}(e) \supset Z(\hat{G})^\Gamma$

$S_e / Z(\hat{G})^\Gamma$ is finite

$\exists$ functor $\mathcal{L}_G \longrightarrow \mathrm{Perv}(\mathrm{Bun}_G \times \overline{\mathbb{F}_p})^{weil}$

$$\begin{array}{ccc} e & \longmapsto & \mathcal{F}_e \\ \circlearrowleft & & \circlearrowleft \\ S_e & = & S_e \end{array}$$

s.t. 1) If $\xi \in X^*(Z(\hat{G})^\Gamma) = \pi_0(\mathrm{Bun}_G / \overline{\mathbb{F}_p})$

and $\mathrm{Bun}_G^\xi \times \overline{\mathbb{F}_p} = X^{-1}(\xi)$

$\mathcal{F}_e |_{\mathrm{Bun}^\xi_{G/\overline{\mathbb{F}_p}}} \circlearrowleft Z(\hat{G})$ acts via $\xi$

2) $\mathcal{F}_e |_{\mathrm{Bun}_G^{s.s.}}$ = local system

$j : \mathrm{Bun}^{\emptyset} \underset{open}{\subset} \mathrm{Bun}$

$\mathcal{F}_e = j_{*}(j^* \mathcal{F}_e)$ when

$e$ elliptic ($e(I_{\mathbb{Q}_p})$ finite)

$\mathcal{F}_e = j_!(j^* \mathcal{F})$

3) $[b] \in B(G)_{basic}$ $\quad x_b : \mathrm{Spa}(\mathbb{C}_p^\flat) \xrightarrow{\mathcal{E}_b} \mathrm{Bun}_{\overline{\mathbb{F}_p}}$

$$x_b^* \mathcal{F}_\varphi = \bigoplus_{\rho \in \widehat{S_e}} \boxed{\mathcal{F}_{\mathcal{C},\rho}} \otimes \rho$$

$S_e \times J_b(\mathbb{Q}_p)$ $\quad \rho_{|Z(\hat{G})^\Gamma} = \chi(b)$

L-packet

$$\mathcal{C} \longmapsto (\mathcal{F}_{\mathcal{C},\rho})_\rho$$

4) Character sheaf ----

5) Hecke property :

$$\mathrm{Bun}_G \xleftarrow{c_1} \mathrm{Hecke}_\mu \xrightarrow{c_2} \mathrm{Bun}_G \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond$$

$$c_{2!}(c_1^* \mathcal{F}_\mathcal{C} \otimes K_\mu) = \mathcal{F}_\mathcal{C} \boxtimes \Omega_\mu \circ \mathcal{C}$$

A few things: * OK for tori. ($\Leftrightarrow$ local class field theory)

* Specialize Hecke at the trivial vector bundle $(x_1^*)$

Find Kottwitz conj. for $H_{et}$ (RZ spaces)

* $G = GL_n$, $\mu = (1, 0, \ldots, 0)$

Scholze conjecture in February about Perverse sheaves pushed forward by $\pi_{HT}$

$G = GL_2$ $\quad b = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}$ $\quad \mathcal{E}_b = \mathcal{O} \oplus \mathcal{O}(1)$

$\mathrm{Aut} = \begin{pmatrix} \mathbb{Q}_p^\times & B_{cris}^{\varphi = p} \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$

$i_b^* \mathcal{F}_b$

If $\ell \neq p$, act trivially.