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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Laurent Clozel

Talk Title: Radical base change mod p

Date: 12 / 5 / 2014 **Time:** 12 : 00 am / pm (circle one)

List 6-12 key words for the talk: Langlands functoriality, base change, mod p cohomology, concrete functoriality

Please summarize the lecture in 5 or fewer sentences: Describes new base change results for mod p cohomology classes building up from a simple example. A key point is the existence of a Brauer map in a general setting between Hecke algebras.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Rad~~icial~~ical base change, mod p

1. 5 ways of doing Langlands functoriality
   (1) Trace formula
   (2) Converse theorem
   (3) Θ - series
   (4) Galois representations and deformation theory
   (5) "Beyond endoscopy"

G, H reductive $/\mathbb{Q}$, ${}^L H \to {}^L G$

$$? \, \varphi_* : \quad \pi_H, \underset{\text{aut.}}{H(\mathbb{A})} \rightsquigarrow \pi_G, \underset{\text{aut.}}{G(\mathbb{A})}$$

$$f_H \text{ on } H(\mathbb{A}) \rightsquigarrow f_G \text{ on } G(\mathbb{A})$$

↑ this is a concrete version of functoriality (e.g. (3))

↳ rationality, etc.

This lecture
(6) $H \subseteq G$? direct construction of $f_H \rightsquigarrow f_G$

$\left\{ \begin{array}{l} \underline{\text{mod } p} \\ \text{concrete} \\ \text{only for forms of cohomological type (cohomology classes)} \end{array} \right.$

2. Base change made easy

Data: B = quaternion algebra$/\mathbb{Q}$ ramified at ∞, p (p odd all throughout)

$F/\mathbb{Q}$ degree p, $F \subseteq F_\infty = \mathbb{Z}_p$-ext. of $\mathbb{Q}$

$$\begin{array}{c} F \\ \big| p \\ \mathbb{Q} \end{array}, \quad F \subseteq \mathbb{Q}(\zeta_{p^2})$$

Base change $G = PGL(1,B)/\mathbb{Q}$ $\qquad G(F)$ Clozel (2)

$K_0 \subseteq G(\mathbb{A}_f)$ $\qquad K_0 = \left(B^\times_{\mathbb{Z}_p}(1) \cdot \prod GL(2,\mathbb{Z}_l)\right)^-$

Mult. group of maximal order, elements $\equiv 1 \bmod p$

+ analogous group at level 1 (with maximal order in new algebra)
$\quad$ 1 = over F

<u>Thus</u>: $S_0 = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K_0$, $\quad S_1$ = defined same way (both finite)

rather think of as tower $\begin{matrix} S_1 \\ | \\ S_0 \end{matrix}$ $\qquad S_0 \hookrightarrow S_1 \quad (p \geq 5)$

$\mathcal{S}_0 = C(S_0, k)$ $\quad \mathcal{S}_1$ same for S_1
$\qquad \qquad \quad \overset{\shortparallel}{\overline{\mathbb{F}}_p}$

$j_* \, \mathcal{S}_0 \to \mathcal{S}_1 \qquad \mathrm{Res}: \mathcal{S}_1 \to \mathcal{S}_0$

Hecke operators: $l \neq p$ inert in $F/\mathbb{Q}$

On S_0: $(T_l f)(\cancel{x}) = \sum_{\xi \in \mathbb{F}_l} f\left(y \begin{pmatrix} l & \xi \\ & 1 \end{pmatrix}\right) + f\left(y \begin{pmatrix} 1 & \\ & l \end{pmatrix}\right)$ inserted at place l

$$\cancel{x} \quad G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K_0^l$$
$$\downarrow$$
$$x \in G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K_0^{\cancel{l}}$$

$\begin{matrix} \lambda \\ | \\ l \end{matrix}$ $\quad$ Also $(T_\lambda f)(x) = \sum \ldots \qquad \xi \in \mathbb{F}_{l^2}$

Finally, the Galois group $\Gamma = \mathrm{Gal}(F/\mathbb{Q})$ acts on S^1 fixing S^0

<u>Basic fact</u> $f \in H^0(\Gamma, \mathcal{S}_1)$ then $\mathrm{Res}_{S_0}(T_\lambda f) = T_l \, \mathrm{Res}_{S_0} f$

Corollary Assume $\eta : \mathbb{T}_0 \to k$ is a (modular) character of Clozel ③
the abstract Hecke algebra for $G/\mathbb{Q}$.

$$\mathbb{T}_0 \twoheadrightarrow k_\eta \quad \text{as a } \mathbb{T}_0\text{-module}$$

Then $\exists$ quotient $\mathbb{T}_1 \twoheadrightarrow k_{\eta_F}$ (η_F obtained from η by composition)
(similar comp. at split prime)

Remark Langlands $a_\ell \mapsto a_\ell^P$
here ℓ split, $a_\ell \mapsto a_\lambda = a_\ell$

③ A theorem of Treumann and Venkatesh

$H \subseteq G$, $H =$ Fixed points of σ, $\sigma^p = 1$

Connected and reductive

Preserved by σ $\rightsquigarrow K \subseteq G(\mathbb{A}_f)$ (decomposes as $\prod K_p$)

$U = K^\sigma \subseteq H(\mathbb{A}_f)$

$$S(G,K) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K$$

$$S(H,U) \equiv S(G,K)^\Sigma \qquad \Sigma = \langle \sigma \rangle$$

① Cohomology $H^\circ(S(G,K), \mathbb{F}_p)$, — for H

② Outside a finite # of primes:
K_ℓ, U_ℓ hyperspecial $\to$ Hecke algebras $\mathcal{H}_G, \mathcal{H}_H$

$\Sigma \circlearrowright \mathcal{H}_G$ so have $\mathcal{H}_G^\Sigma$

<u>Fact</u> $\exists$ natural map (of algebras): $\mathcal{H}_G^{\Sigma} \to \mathcal{H}_H$ (for good primes)

Thus: $$Br: (\mathcal{H}_G^S)^{\Sigma} \to (\mathcal{H}_H^S)$$
$$(\mathcal{H}_G^S)^{\Sigma} = \bigotimes_{v \in S} \mathcal{H}_v^{\Sigma}$$

<u>Theorem 1</u> Assume $\eta_H : \mathcal{H}_H^S \to k$ occurs in the cohomology of $S(H, U)$. Then $\eta_H \circ Br$ occurs in $H^{\bullet}(S(G, K))$

(4) Automorphic induction

$F = \#$Field, $G = GL(n, F)$

<u>Recall</u> $v =$ prime, $\mathcal{H}_v$ unramified $v \nmid p$

(1) $\mathcal{H}_v \otimes k \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{G_n}$

$$S_\alpha = \sum_{|I| = \alpha} \prod_{i \in I} x_i$$

$$T_\alpha = K \begin{pmatrix} \varpi & & & 0 \\ & \ddots & & \\ & & \varpi & \\ 0 & & & 1 \end{pmatrix} K \mapsto q_v^{\frac{\alpha(1-\alpha)}{2}} S_\alpha$$

(2) $\eta_v : \mathcal{H}_v \to k \Rightarrow (t_1, \dots, t_n) \in (k^\times)^n$

Also unram. semisimple rep of W_{F_v} s.t. $\mathrm{Frob}_v \mapsto (t_1, \dots, t_n)$

(3) <u>Conjecture</u> (Ash) If $\eta : H^S \to k$ occurs in cohomology, $\exists$ a semisimple rep r of $Gal(\bar{F}/F)$ s.t. for $v \notin S$, $\eta_v \leftrightarrow r(\mathrm{Frob}_v)$

(4) If F is a CM field, r exists

(Harris-Lan-Taylor-Thorne for classes lifting to char. 0, Scholze in general)

Now, $F = \#$ field, $E = F(\sqrt[p]{1})$ for $\zeta \notin F^p$

Let $\chi : A_E^\times / E^\times \to \mathcal{K}$

$(\chi : G_E \to \mathcal{K})$

<u>Theorem A</u> : $\exists\ K \subseteq GL(p, \mathbb{A}_F^\infty)$, $w \in H^0(S(G,K), \mathcal{K})$ eigenclass for $\mathcal{H}_G^S$

with character $\eta : \mathcal{H}_G^S \to \mathcal{K}$ s.t. η is associated to $\mathrm{ind}_{G_E}^{G_F} \chi$

More general case $[E/F] = p$

$H = GL(M, E) \hookrightarrow GL(N, F)$ $\quad N = Mp$

<u>Theorem B</u> : Assume $\eta_E : \mathcal{H}_H^S \to \mathcal{K}$ occurs in the cohomology of $S(H, U)$. Then $\exists\ \eta_F : \mathcal{H}_G^S \to \mathcal{K}$ s.t. η_F occurs in $H^0(S(G,K), \mathcal{K})$ and for any ~~prime~~ place v of E, $v \notin S$,

$$r(\eta_{F,v}) = \bigoplus_{w | v} \mathrm{ind}_{W_E}^{W_F} r(\eta_{E,w})$$

<u>Theorem C</u> : Assume F is CM, linearly disjoint from $\mathbb{Q}(\zeta_p)$. Then, if η_E is a cohom. character of $GL(M, E)$, there is an Mp-dimensional representation R of G_E such that

$$R = r_E \oplus \bigoplus_{0}^{p-2} r_E \otimes \omega^i \quad \text{where } \omega = \text{mod } p \text{ cyclo}$$

(identity of local rep's almost everywhere).