MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Remarks the Cohomology of the Lubin-Tate Tower - Peter Scholze

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Summary: The speaker describes a construction that takes an admissible $\mathbb{F}_p$-representation of a p-adic field and obtains an admissible $D^\times$-representation in it from cohomology of a sheaf on the infinite-level Lubin-Tate tower that is descended to projective space via the Gross-Hopkins period map. The proof is explained by showing how it is modeled off of a proof of a finiteness theorem from a previous lecture. The speaker then discusses local-global compatibility for these representations.

Originally the talk was supposed to be "Future directions II: Local Langlands and equivariant sheaves on projective space", but the speaker was asked by many participants to speak on this topic instead.

Setup: Let $F/\mathbb{Q}_p$ be a finite extension of degree $n \geq 1$. Let $\mathcal{O}_F$ be the ring of integers, ϖ a uniformizer, $k = \mathcal{O}/\varpi$ the residue field, and $\overline{k}$ an algebraic closure of it. Fix $\breve{F}$ to be the completion of the unramified extension of F with residue field $\overline{k}$. From Weinstein's talks, we have the Lubin-Tate space $\mathcal{M}_{LT,\infty}$ (which was a perfectoid space) on which $\mathrm{GL}_n(F) \times D^\times$ acts, where D/F is the division algebra of invariant $1/n$. Also we have the Gross-Hopkins period map $\pi_{GH} : \mathcal{M}_{LT,\infty} \to \mathbb{P}^{n-1}_{\breve{F}}$. This is $D^\times$-equivariant (for the natural action of $D^\times$ on $\mathbb{P}^{n-1}$) and also $\mathrm{GL}_n(F)$-equivariant (for the trivial action on $\mathbb{P}^{n-1}$).

Theorem 1 (Gross-Hopkins)**.** *The map π_{GH} is surjective, so is a $\mathrm{GL}_n(F)$-torsor.*

Remark: The surjectivity here is absolutely crucial to what we're doing, and the argument won't carry over to other Rapoport-Zink spaces because the period maps there aren't surjective!

Recall some facts about ℓ-adic cohomology for $\ell \neq p$. Fix a supercuspidal representation π of $\mathrm{GL}_n(F)$. Then:

Theorem 2 (Harris-Taylor, Mieda)**.** *Take $C/\breve{F}$ algebraically closed and complete. Consider*

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, H^i_c(\mathcal{M}_{LT,\infty,C}, \overline{\mathbb{Q}}_\ell)).$$

This still has an action of $D^\times$, and also an action of W_F (extending the action of inertia via Weil descent, coming from the Galois group of C). Then, as a $D^\times \times W_F$-module, this space is isomorphic to $\mathrm{JL}(\pi)\otimes\mathrm{LLC}(\pi)$ (up to some twists and duals) if $i = n-1$, and is trivial otherwise.

One would like to have a similar result in the p-adic case. But then we run into a problem: there's no finiteness results for the $\mathbb{F}_p$-cohomology. Even at finite level, the cohomology changes if we change the algebraic closure C. However, we'll see that if one does things in the correct way we still get a finiteness result.

Construction: Let π be an admissible $\mathbb{F}_p$-representation of $\mathrm{GL}_n(F)$ on a vector space V. We descend the constant sheaf $\underline{V}$ over $\mathcal{M}_{LT,\infty}$ to $\mathbb{P}^{n-1}_{\breve{F}}$ via the $\mathrm{GL}_n(F)$-action. Get a sheaf $\mathcal{F}_\pi$ on $\mathbb{P}^{n-1}_{\breve{F},\eta}$.

Theorem 3. *For all $i \geq 0$, the group $H^i(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi)$ (which has a natural action of $D^\times \times W_F$) is an admissible $D^\times$-representation, is independent of C, and zero for $i > 2(n-1)$.*

Proposition 4. *This is compatible with global correspondences.*

Strategy for proving the finiteness theorem: follow the proof of the "Old Theorem" that Nizioł explained, that if X/C is proper and smooth then $H^i(X_{\text{ét}}, \mathbb{F}_p)$ is finite-dimensional. There were two main steps:
(1) Prove almost-finite-generation of $H^i(X_{\text{ét}}, \mathcal{O}_X^+/p)$.
(2) Use Artin-Schreier sequence argument to get an almost-isomorphism

$$H^i(X_{\text{ét}}, \mathbb{F}_p) \otimes \mathcal{O}_C/p \;\cong_a\; H^i(X_{\text{ét}}, \mathcal{O}_X^+/p).$$

Of course, in our new case we don't want a finite-dimensional representation, but an admissible one, so need to change our setup a bit. To do this we define a funny cohomology theory.

Definition 5. Fix $K \subseteq D^\times$ a compact open. If $\mathcal{G}$ is a $D^\times$-equivariant sheaf on $\mathbb{P}^{n-1}$, define a cohomology group

$$R\Gamma(\mathbb{P}^{n-1}_C/K, \mathcal{G}) = R\Gamma_{\text{cont}}(K, R\Gamma(\mathbb{P}^{n-1}_C, \mathcal{G})).$$

The notation is because we want to think of descending $\mathcal{G}$ to a sheaf on a quotient $\mathbb{P}^{n-1}_C/K$, but this doesn't quite make sense itself. Then, we have the following key proposition.

Proposition 6. *$H^i(\mathbb{P}^{n-1}_C/K, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$ is almost finitely generated.*

If we assume this, then step 2 of the argument above goes through, and we conclude:

Corollary 7. *The group $H^i(\mathbb{P}^{n-1}_C/K, \mathcal{F}_\pi)$ is finite-dimensional, and we have an almost-isomorphism*

$$H^i(\mathbb{P}^{n-1}_C/K, \mathcal{F}_\pi) \otimes \mathcal{O}_C/p \;\cong_a\; H^i(\mathbb{P}^{n-1}_C/K, \mathcal{F}_\pi \otimes \mathcal{O}^+/p).$$

Thus if we take a direct limit over K, get the "basic comparison theorem"

$$H^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi) \otimes \mathcal{O}_C/p \;\cong_a\; H^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p).$$

Corollary 8. *$H^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)$ is an admissible $D^\times$-representation.*

Proof. Induct on i (so assume the result holds for all degrees $i' < i$). Then there's a Hochschild-Serre spectral sequence

$$H^{m_1}_{\mathrm{cont}}(K, H^{m_2}(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)) \implies H^{m_1+m_2}(\mathbb{P}_C^{n-1}/K, \mathcal{F}_\pi).$$

Then, it's a fact that if ρ is an admissible $D^\times$-representation then the dimension of all $H^j_{\mathrm{cont}}(K, \rho)$ are finite. Then, if we look at the terms contributing to $H^i(\mathbb{P}_C^{n-1}/K, \mathcal{F}_\pi)$, there are a bunch of terms with $m_1 < i$ (which are finite-dimensional by induction) and a term $H^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)^K$. Since $H^i(\mathbb{P}_C^{n-1}/K, \mathcal{F}_\pi)$ is finite-dimensional by the above corollary, this forces $H^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)^K$ to be finite-dimensional. □

So we need to prove the key proposition. Back to the "old theorem": we had X/C proper and smooth, and we use an argument of shrinking covers due to Cartan-Serre and Kiehl. The idea is to take finite covers

$$X = \bigcup_{i \in I} U_i = \bigcup_{i \in I} V_i$$

with U_i, V_i affinoids satisfying $\overline{U}_i \subseteq V_i$ (and having good coordinates, etc.). Then the key lemma was:

Lemma 9. *Let U, V be affinoids of finite type over C with $\overline{U} \subseteq V$. Then $H^i(V_{\text{ét}}, \mathcal{O}^+/p) \to H^i(U_{\text{ét}}, \mathcal{O}^+/p)$ has almost-finitely-generated image.*

Now, we turn back to the new case of our key proposition,. Take finite covers

$$\mathbb{P}_C^{n-1} = \bigcup_{i \in I} U_i = \bigcup_{i \in I} V_i$$

with U_i, V_i affinoids satisfying $\overline{U}_i \subseteq V_i$ (and the other properties we needed above). Moreover can assume the U_i and V_i are K-stable by shrinking K if need be. Now, since $\pi_{GH} : \mathcal{M}_{LT,0,C} \to \mathbb{P}_C^{n-1}$ is surjective, the inclusion $V_i \to \mathbb{P}_C^{n-1}$ lifts to a map $V_i \to \mathcal{M}_{LT,0,C}$. Also, note that $\mathcal{F}_\pi|_{\mathcal{M}_{LT,0}}$ depends only on $\pi|_{\mathrm{GL}_n(\mathcal{O}_F)}$, as $\mathcal{M}_{LT,\infty} \to \mathcal{M}_{LT,0}$ is a $\mathrm{GL}_n(\mathcal{O}_F)$-torsors.

Lemma 10. *If $U, V \subseteq \mathcal{M}_{LT,0,C}$ are K-stable affinoids with $\overline{U} \subseteq V$, then for any admissible $\mathrm{GL}_n(\mathcal{O}_F)$-representation π, the image of*

$$H^i(V/K, \mathcal{F}_\pi \otimes \mathcal{O}^+/p) \to H^i(U/K, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$$

is almost finitely generated.

Proof. We start by taking a resolution of π by a complex whose terms are finite products of $C(\mathrm{GL}_n(\mathcal{O}_F), \mathbb{F}_p)$. Then there's a spectral sequence computing the cohomology of π in terms of the cohomology of the resolution, so we can reduce to the case where $\pi = C(\mathrm{GL}_n(\mathcal{O}_F), \mathbb{F}_p)$.

So have $U \subseteq V \subseteq \mathcal{M}_{LT,0,C}$. Can take the preimages under the map $f : \mathcal{M}_{LT,\infty,C} \to \mathcal{M}_{LT,0,C}$, giving $U_\infty \subseteq V_\infty$ with $\overline{U}_\infty \subseteq V_\infty$. Moreover, $\mathcal{F}_\pi = f_* \mathbb{F}_p$, so we conclude

$$H^i(V/K, \mathcal{F}_\pi \otimes \mathcal{O}^+/p) = H^i(V_\infty/K, \mathcal{O}^+/p).$$

Next, we use the isomorphism between the Lubin-Tate tower and the Drinfeld tower, so we can move U_∞ and V_∞ over to $\mathcal{M}_{Dr,\infty,C}$. But now, since $K \subseteq D^\times$ is compact open, we can pass to a finite level $\mathcal{M}_{Dr,K,C}$ which is locally finite-type over C, and get affinoids U_K, V_K with $\overline{U}_K \subseteq V_K$.

Finally, it's obvious that

$$H^i(V_\infty/K, \mathcal{O}^+/p) = H^i(V_K, \mathcal{O}^+/p).$$

So we're reduced to showing that

$$H^i(V_K, \mathcal{O}^+/p) \to H^i(U_K, \mathcal{O}^+/p)$$

has finitely-generated image. But this follows from the lemma mentioned above. □

Local-global compatibility: Let $\mathbb{F}^+$ be a totally real field and $\mathbb{F}/\mathbb{F}^+$ be a CM extension. Suppose that there's only one place over p, that the corresponding localization $(\mathbb{F}^+)_p$ is isomorphic to our local field F from above, and that $\mathbb{F}/\mathbb{F}^+$ is split at p. Take $G/\mathbb{F}^+$ a compact unitary group which is GL_n at p. Fix $K^p \subseteq G(\mathbb{A}^p_{\mathbb{F}^+,f})$. Then, let

$$\pi = C(G(\mathbb{F}^+) \backslash G(\mathbb{A}^p_{\mathbb{F}^+,f})/K^p, \overline{\mathbb{F}}_p),$$

which has an action of $\mathrm{GL}_n(F)$ and also of a Hecke algebra $\mathbb{T}_{K^p}$ away from p.

The question is then, what happens if we plug in this π to the machine above? For this, look at an inner form G' of G (which is now $D^\times$ at p, and $U(1, n-1)$ at some infinite place, and left the same as G at the other places). This gives rise to a compact Shimura variety Sh_{K^p}. Can then look at

$$\pi' = H^i(\mathrm{Sh}_{K^p}, \overline{\mathbb{F}}_p),$$

which has an action of $D^\times \times \mathrm{Gal}_{\mathbb{F}}$ (though may have needed to use a similitude group to get this Galois action). We then have:

Proposition 11. *We have $H^i(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi) \cong \pi'$ as representations of $D^\times \times \mathrm{Gal}_F \times \mathbb{T}_{K^p}$.*

Now, fix a maximal ideal $\mathfrak{m} \subseteq \mathbb{T}_{K^p}$. By a bunch of big theorems, we know that there exists a $\rho_{\mathfrak{m}} : \mathrm{Gal}_{\mathbb{F}} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ which has the correct Hecke eigenvalues and $\rho_{\mathfrak{m}}|_{\mathrm{Gal}_F}$ reducible. Assume some sort of big image condition, say that $\operatorname{img} \rho_{\mathfrak{m}} = \mathrm{GL}_n(\mathbb{F}_{p^r})$ for some r. Then, claim that $H^i(\mathbb{P}_C^{n-1}, \mathcal{F}_{\pi_{\mathfrak{m}}})^K$ is $\rho_{\mathfrak{m}}|_{\mathrm{Gal}_F}$-isotypic (and not all zero).

Key input (that requires the big image assumption): A theorem of Emerton-Gee that $H^i(\mathrm{Sh}_{K^p}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ is $\rho_{\mathfrak{m}}$-isotypic. When one knows this, the above is immediate from the proposition (except the "not all zero" part, but that's not hard to show).