Mathematical Preparation of Teachers

MSRI

March 26, 2014

H. Wu
An ad by IBM in London’s Heathrow Airport (March 2008):

**Stop** selling what you have.

**Start** selling what they need.
For the mathematical preparation of teachers:

Our universities have been too busy selling what they have.
For the mathematical preparation of teachers:

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They don't think about what pre-service teachers need.
What do they need?
Pedagogical knowledge, definitely.
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*We have let them down.*
The mathematics that has been taught in schools for more or less the past four decades is what we call **TSM, Textbook School Mathematics**.

TSM is what school textbooks have in common overall: almost no definitions, fragmented presentation of sound bites, blurring the line between a proof and a heuristic argument, and lack of precision.

In other words, *not learnable*. 
Consequences of TSM:

1. (2011 TIMSS, 8th grade) \( \frac{1}{3} - \frac{1}{4} = ? \)

32% of U.S. students chose \( \frac{1 - 1}{4 - 3} \).

26% chose \( \frac{1}{4 - 3} \).

30% got it right. (Taipei: 82%. Finland: 16%).
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(Do they try to make sense of anything at all?)
2. Many (most?) high school students believe that \( \frac{-7}{-3} = \frac{7}{3} \) because:

they are told that \( \text{neg \times neg = pos} \), therefore it is reasonable that \( \text{neg \div neg = pos} \).
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they are told that \(\text{neg} \times \text{neg} = \text{pos}\), therefore it is reasonable that \(\text{neg} \div \text{neg} = \text{pos}\).

(Do they try to reason abstractly?)
3. (Division of fractions) Students are taught $32 \div 5 = 6 \ R 2$, therefore $5 \div \frac{3}{4} = 6 \ R \ \frac{1}{2}$;

They guess that $\frac{1}{2} = \frac{2}{3} \times \frac{3}{4}$. Therefore $5 \div \frac{3}{4} = 6\frac{2}{3}$. 

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(What to do for the division $\frac{2}{11} \div \frac{81}{29}$? How can they critique this reasoning?)
4. The following table gives the number of miles Helena runs in minutes:

<table>
<thead>
<tr>
<th>min</th>
<th>mi</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>30</td>
<td>3</td>
</tr>
</tbody>
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How many miles does she run in 25 min?
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<tbody>
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How many miles does she run in 25 min?

Students learn to model the data by proportional reasoning. The unit rate is \( \frac{1}{10} \) mi/min. So in 25 minutes she runs \( 25 \times \frac{1}{10} = 2\frac{1}{2} \) miles.
But it turns out that this is an Olympic 400 meter specialist training for a meet. Every 10 minutes, she runs \( \frac{1}{2} \) mile in 2 minutes and walks the next \( \frac{1}{2} \) mile in 8 minutes. So in 25 minutes, she covers about 2.7 miles.
But it turns out that this is an Olympic 400 meter specialist training for a meet. Every 10 minutes, she runs $\frac{1}{2}$ mile in 2 minutes and walks the next $\frac{1}{2}$ mile in 8 minutes. So in 25 minutes, she covers about 2.7 miles.

(Why can’t proportional reasoning be used to model this situation?)
5. Students are convinced that for all positive $a, b,$

$$\sqrt{a} \sqrt{b} = \sqrt{ab},$$

because, on the calculator,

$$\sqrt{5} \sqrt{7} = \sqrt{35} = 5.9160797831,$$

$$\sqrt{3} \sqrt{6} = \sqrt{18} = 4.24264068712,$$ etc.
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$\sqrt{3} \sqrt{6} = \sqrt{18} = 4.24264068712,$ etc.

(Isn’t this a strategic use of the calculator?)
6. Because *similar* means same shape but not necessarily the same size, students believe that the following curves are not similar.
They also believe that the left curve above is similar to the following curve:
Turns out the first two curves are graphs of \( x^2 + 10 \) and \( \frac{1}{360} x^2 + 10 \), resp., and are therefore similar.

The third curve is the graph of \( \frac{1}{4} x^4 + x^2 + 1 \), and is therefore not similar to the first curve.
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The third curve is the graph of $\frac{1}{4} x^4 + x^2 + 1$, and is therefore not similar to the first curve.

\textit{(Perhaps we need a precise definition of similarity?)}
7. Students just learn about equivalent fractions. They learn that \( \frac{2}{3} = \frac{8}{12} \), because,

\[
\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}.
\]

The following conversation then takes place:
Carl: You know, I have thought about it, and I don’t know why \( \frac{2}{3} \times \frac{5}{5} = \frac{2 \times 5}{3 \times 5} \).

Bryant: Look, you see 2 and 5 on top with \( \times \) in between, and you multiply. The same with 3 and 5. You know how it is with whole numbers, right?

Carl: Is that how you do it? So \( \frac{2}{3} + \frac{5}{5} = \frac{2+5}{3+5} \)?

Diane: Great! Now we can add fractions too!
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Diane: Great! Now we can add fractions too!

(But is this the right way to make use of structure?)
8. Students learn about why \((-2) \cdot (-5) = 10\) by observing regularity in repeated reasoning:

\[
\begin{align*}
3 \cdot (-5) &= -15 \\
2 \cdot (-5) &= -10 \\
1 \cdot (-5) &= -5 \\
0 \cdot (-5) &= 0 \\
(-1) \cdot (-5) &= ? \\
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The right side increases by 5 when going down each step, so the last two lines have to be 5 and 10.
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(This is how they will learn algebra?)
Now look at the life-cycle of a math teacher:

In K–12 learns TSM.

→ In college learns advanced math or more TSM, and strategies to implement what they know about TSM.

→ Teaches in K–12 by regurgitating TSM.

→ Victimizes the next generation of teachers by teaching them TSM.
In the fall, teachers will be asked to implement the CCSSM.

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Equipped only with a knowledge of TSM, teachers have little hope of implementing the CCSSM.
If a general sends soldiers to the front without any ammunition, he would be court-martialed, at least.
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Yet, universities can do this to prospective teachers year after year with impunity.

This is not something the math departments—in fact the math community—should be proud of.
Two remarks:

(1) Why not get rid of TSM by writing reasonable *textbooks*?
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(1) Why not get rid of TSM by writing reasonable textbooks?

(2) If we in the math departments continue this tradition of inaction in teaching prospective teachers correct school mathematics, we will victimize not only teachers, but math educators as well.