A Bachelor of Science in Mathematics that Emphasizes Mathematical Meanings for Teaching Mathematics

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Project Aspire: Defining and Assessing Mathematical Knowledge for Teaching Secondary Mathematics

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Katherine Castellano, Post Doc
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Ben Whitmire, URE

DUE-1050595
A Problem

- Students leave high school with poorly formed meanings for ideas of the secondary mathematics curriculum
- Students take mathematics courses that presume they have meanings they in fact do not have
- Students apply coping mechanisms in college math that allowed them to succeed in high school mathematics (memorization)
- Students return to high schools to teach ideas they understood poorly, rarely revisited, and for which they still have poorly-formed meanings
Overview

- Illustrate problems with mathematical meanings drawn from a survey of 260 high school mathematics teachers
- Draw implications for undergraduate mathematics education and for teacher preparation
- Describe ASU’s Bachelor of Science in Mathematics with math education concentration
  - Focus on Mathematical Meanings for Teaching secondary mathematics
  - Focus on ways to lead high school students’ development of coherent mathematical meanings and ways of thinking
Teachers’ Mathematical Meanings
MMTsm
(Mathematical Meanings for Teaching secondary mathematics)

- A 43-item diagnostic instrument: Six animated items, nine multiple choice, 28 free-response items
- Covers ideas of function, function notation, functions as models, equation, quantitative reasoning, rate of change, proportionality, frames of reference, variation and covariation, and representational equivalence
- Administered to 160 high school teachers (summer 2012), 100 high school teachers (summer 2013). Teachers drawn from southwest and midwest U.S.
- Reliability and validity being established in conjunction with BEAR group at UC Berkeley.
Attending to Mathematical Meanings

A college science textbook contains this statement about a function $f$ that gives a bacterial culture’s mass at moments in time.

*The change in the culture’s mass over the time period $\Delta x$ is 4 grams.*

**Part A.** What does the word “over” mean in this context?

“During”

**Part B.** Express the textbook’s statement in mathematical notation.

$$f(x + \Delta x) - f(x) = 4$$
Attending to Mathematical Meanings

The change in the culture’s mass over the time period $\Delta x$ is 4 grams.

Part A. What does the word “over” mean in this context?

$\Delta x$ is the bottom part of the ratio.

Part B. Express the textbook’s statement in mathematical notation.

$$\frac{\text{mass}}{\Delta x} = 4$$
Attending to Mathematical Meanings

The change in the culture’s mass over the time period $\Delta x$ is 4 grams.

Part A. What does the word “over” mean in this context?

$m$ is dependent on time ($t$)

Part B. Express the textbook’s statement in mathematical notation.

$\frac{m}{t}$
The change in the culture’s mass over the time period $\Delta x$ is 4 grams.

**Part A.** What does the word “over” mean in this context?

<table>
<thead>
<tr>
<th>Level A0</th>
<th>The teacher wrote, “I don’t know”, the scorer cannot interpret the teacher’s response, or the response did not address the question.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level A1</td>
<td>The response conveys the idea that “over” means division</td>
</tr>
</tbody>
</table>
| Level A2 | • The teacher wrote both “divide” and “during” as meanings for “over”, or  
• The teacher wrote “during” but described the time interval as something other than the stated interval, or  
• The teacher alluded to a passage of time, but without directly stating an interpretation of the word “over” |
| Level A3 | The teacher wrote “during” or something equivalent                                                                      |

**Part B.** Express the textbook’s statement in mathematical notation.
### Part B. Express the textbook’s statement in mathematical notation.

| Level B0 | The teacher wrote “I don’t know” or reworded the original sentence  
|          | The scorer cannot interpret the teacher’s response, or  
|          | The response contained a mathematical expression that is not described in Levels B1-B3. |
| Level B1 | The teacher wrote a quotient showing the change in mass divided by the change in $x$ is equal to 4, or some rearrangement of that statement. |
| Level B2 | The teacher wrote a true, but vague, statement such as $\Delta m = 4$  
|          | The response contained a combination of the idea of $\Delta m = 4$ and the notation $\Delta m/\Delta x = 4$ |
| Level B3 | Teacher wrote something equivalent to $f(x + \Delta x) - f(x) = 4$ |
Attending to Mathematical Meanings

The change in the culture’s mass over the time period $\Delta x$ is 4 grams.

Part A. What does the word “over” mean in this context?

Part B. Express the textbook’s statement in mathematical notation.

<table>
<thead>
<tr>
<th></th>
<th>B0</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>BX</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A0</td>
<td>8</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>A1</td>
<td>15</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>A2</td>
<td>29</td>
<td>16</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>52</td>
</tr>
<tr>
<td>A3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>AX</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>54</td>
<td>37</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>100</td>
</tr>
</tbody>
</table>
The change in the culture’s mass over the time period $\Delta x$ is 4 grams.

**Part A.** What does the word “over” mean in this context?

**Part B.** Express the textbook’s statement in mathematical notation.

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<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>10</td>
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<td>3</td>
<td>1</td>
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</tr>
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<td>AX</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>total</td>
<td>54</td>
<td>37</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>100</td>
</tr>
</tbody>
</table>

54 of 62 teachers (87%) who said that “over” means “during” wrote nonsense or an expression involving division by $\Delta x$. 
The change in the culture’s mass over the time period $\Delta x$ is 4 grams.

**Part A.** What does the word “over” mean in this context?

**Part B.** Express the textbook’s statement in mathematical notation.
The change in the culture’s mass over the time period $\Delta x$ is 4 grams.

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The change in the culture’s mass over the time period $\Delta x$ is 4 grams.

**Part A.** What does the word “over” mean in this context?

**Part B.** Express the textbook’s statement in mathematical notation.
Why is “Over” Important?

- “Over” is not important
- But the orientation to computational interpretations that it reveals is important
Function Notation

The functions $f$, $g$, and $h$ are defined below.

\[
f(u) = u^2 - 1
\]
\[
g(s) = 1 + \frac{f(2s + 1)}{2}
\]
\[
h(r) = g(r) - 1
\]

What is $h(2)$? Show your work.

It seems there is not a major problem with teachers’ meanings for function notation. However ...

<table>
<thead>
<tr>
<th></th>
<th>Math</th>
<th>MathEd</th>
<th>Other</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incorrect</td>
<td>4</td>
<td>4</td>
<td>11</td>
<td>19</td>
</tr>
<tr>
<td>Correct</td>
<td>20</td>
<td>28</td>
<td>17</td>
<td>65</td>
</tr>
<tr>
<td>No Ans</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>total</td>
<td>24</td>
<td>32</td>
<td>29</td>
<td>85</td>
</tr>
</tbody>
</table>
Function Notation (Summer 2012)

Here are two function definitions.

\[ w(u) = \sin(u - 1) \text{ if } u \geq 1 \]

\[ q(r) = \sqrt{r^2 - r^3} \text{ if } 0 \leq r < 1 \]

Here is a third function \( c \), defined in two parts, whose definition refers to \( w \) and \( q \). Place the correct letter in each blank so that the function \( c \) is properly defined.

\[
\begin{align*}
  c(\nu) = & \begin{cases} 
    q(\underline{\nu}) \text{ if } 0 \leq \underline{\nu} < 1 \\
    w(\underline{\nu}) \text{ if } \underline{\nu} \geq 1 
  \end{cases}
\end{align*}
\]
Function Notation (Summer 2012)

Here is a third function \( c \), defined in two parts, whose definition refers to \( w \) and \( q \). Place the correct letter in each blank so that the function \( c \) is properly defined.

\[
c(v) = \begin{cases} 
q(r) & \text{if } 0 \leq r < 1 \\
w(u) & \text{if } u \geq 1 
\end{cases}
\]

Here are two function definitions.

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w(u) = \sin(u - 1) \text{ if } u \geq 1
\]

\[
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\]

<table>
<thead>
<tr>
<th>Response</th>
<th>Math</th>
<th>MathEd</th>
<th>Other</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>R U</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>19</td>
</tr>
<tr>
<td>V</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>Mixed — V, R, and U</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>I don’t know (or equivalent) — teacher’s words</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>No answer – nothing written</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
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<td>total</td>
<td>8</td>
<td>16</td>
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Here is a third function $c$, defined in two parts, whose definition refers to $w$ and $q$. Place the correct letter in each blank so that the function $c$ is properly defined.

$$c(v) = \begin{cases} q(\_\_) & \text{if } 0 \leq \_\_ < 1 \\ w(\_\_) & \text{if } \_\_ \geq 1 \end{cases}$$

<table>
<thead>
<tr>
<th>Response (Taught $\geq$ Precalculus)</th>
<th>Math</th>
<th>MathEd</th>
<th>Other</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>R U</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>V</td>
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<td>3</td>
</tr>
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<td>1</td>
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<td>0</td>
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\[ c(v) = \begin{cases} 
q(\_\_) \text{ if } 0 \leq \_\_ < 1 \\
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\end{cases} \]

Consistent with thinking about function notation idiomatically. “$w(u)$” is the function’s name.

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<td>2</td>
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<td>0</td>
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<tr>
<td>No answer – nothing written</td>
<td>0</td>
<td>2</td>
<td>0</td>
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</tr>
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\end{cases}
\]

**Part B**

James, a student in an Algebra 2 class, defined a function \( f \) to model a situation involving the number of possible unique handshakes in a group of \( n \) people. He defined \( f \) as:

\[
f(n) = \frac{n(n+1)}{2}
\]

According to James’ definition, what is \( f(9) \)?
Part B
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According to James’ definition, what is \( f(9) \)?

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<tr>
<th></th>
<th>Math</th>
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<th>Other</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0</td>
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<td>12</td>
<td>23</td>
<td>47</td>
</tr>
<tr>
<td>Level 1</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>Level 2</td>
<td>10</td>
<td>21</td>
<td>10</td>
<td>41</td>
</tr>
<tr>
<td>Level X</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>total</td>
<td>25</td>
<td>37</td>
<td>36</td>
<td>98</td>
</tr>
</tbody>
</table>
Distinction Between Input and Argument (Summer 2012)

The function $h$ is strictly increasing, and $h(b - 5) = 9$ for some number $b$.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$? Explain.
Distinction Between Input and Argument (Summer 2012)

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<td>2</td>
<td>7</td>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>Function name as multiplication</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>No answer -- nothing written</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$9 - 5 = 9$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>I don’t know</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
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Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x)$? Explain.

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The function $h$ is strictly increasing, and $h(b - 5) = 9$ for some number $b$.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$?

Explain.

<table>
<thead>
<tr>
<th>Response</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(b, 9)$</td>
<td></td>
<td></td>
<td></td>
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<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>I don’t know</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>total</td>
<td>10</td>
<td>17</td>
<td>14</td>
<td>41</td>
</tr>
</tbody>
</table>

For $(b, 9)$, $9 = h(b - 5)$

Thus $h = \frac{9}{b - 5}$

$h(b - 5) = 9$

$\frac{9}{b - 5} = 9$

$9 = 9$

BS Math Ed
Taught Precalc >10 times
Taught Calc AB >15 times
Distinction Between Input and Argument
(Summer 2013, 87 HS Math Teachers)

The function $h$ is strictly increasing, and $h(b - 5) = 9$ for some number $b$.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$?

Explain.

1. $b$ is the input that gives 9 as the output so the point $(b, 9)$ is on the graph.
2. $(b, 9)$ is on the graph because the function is strictly increasing and the input to $h$ is $b - 5$.
3. Solve for $h$ in $h(b - 5) = 9$. When $b$ is 6, then $h$ is 9. So $(b, 9)$ is on the graph.
4. $b - 5$ is in the $x$ position and 9 is in the $y$ position so the $(x, y)$ point is $(b - 5, 9)$.
5. $b - 5$ is the input that gives 9 as the output so the point $(b - 5, 9)$ is on the graph.

Results were Independent of Major
Distinction Between Input and Argument
(Summer 2013, 87 HS Math Teachers)

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Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$? Explain.

1. $b$ is the input that gives 9 as the output so the point $(b, 9)$ is on the graph.
2. $(b, 9)$ is on the graph because the function is strictly increasing and the input to $h$ is $b - 5$.
3. Solve for $h$ in $h(b - 5) = 9$. When $b$ is 6, then $h$ is 9. So $(b, 9)$ is on the graph.
4. $b - 5$ is in the $x$ position and 9 is in the $y$ position so the $(x, y)$ point is $(b - 5, 9)$.
5. $b - 5$ is the input that gives 9 as the output so the point $(b - 5, 9)$ is on the graph.

Results were Independent of Major

- 26% 1.
- 3% 2.
- 6% 3.
- 7% 4.
- 42% 5.
Interpreting Statements Quantitatively

Every second, Julie travels \( j \) meters on her bike and Stewart travels \( s \) meters by walking, where \( j > s \). In any given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel \( j - s \) meters more than Stewart.
b. Julie will travel \( j \cdot s \) meters more than Stewart.
c. Julie will travel \( j/s \) meters more than Stewart.
d. Julie will travel \( j/s \) times as many meters as Stewart.
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<th>Math Ed</th>
<th>Other</th>
<th>Total</th>
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<td><strong>Total</strong></td>
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<td>37</td>
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Meaning in Undergraduate Mathematics

- Assessment covered ideas of function, function notation, functions as models, rate of change, quantitative reasoning, equation, proportionality, frames of reference, variation, and representational equivalence.

- In all areas, a majority of teachers who took substantial undergraduate mathematics from mathematics departments showed weak meanings like those demonstrated today.

- It seems reasonable to conclude that these teachers had these weak meanings while enrolled in their undergraduate math courses.
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We are now testing this
ASU’s Response
Bachelor of Sciences in Mathematics with concentration in Math Ed

- Bachelor of Science in Mathematics
- Graduates with this degree are licensed by AZ Dept of Ed to teach secondary mathematics
- Focused “like a laser” on future teachers’ mathematical meanings for teaching secondary mathematics and on students’ mathematics
- Specialized courses in mathematics education in the math department
- Faculty in Math Ed: Pat Thompson, Marilyn Carlson, Luis Saldanha, Kyeong Hah Roh, Carla Van de Sande, Fabio Milner, Mark Ashbrook, Stacy Musgrave
Specialized Courses in Math Education (Mathematics Department)

- Algebra and Geometry in the High School (1st semester)
- Mentored Tutoring
- Calculus developed according to Harel’s *Necessity Principle* (1st semester)
- Technology and Mathematical Visualization (3rd semester)
- Mathematics Curriculum and Assessment in Grades 7-12 (5th semester)
- Development of Mathematical Thinking (7th semester)
- Methods of Teaching Secondary Mathematics (7th semester)
Algebra and Geometry in the High School (1st semester)

- Central meanings in high school mathematics and how they can be built coherently (Marilyn Carlson’s Pathways curriculum)
- Six hours of tutoring per week in ASU’s precalculus tutoring center
- Coordination between course and tutoring

Start the process of building images of others’ mathematics
Technology and Mathematical Visualization (3rd semester)

- Creating didactic objects — artifacts that are designed to support reflective conversations about important mathematical ideas and ways of thinking
- Students must use mathematics to create them
- Focus simultaneously on future teachers’ mathematics and their future students’ mathematics
- Emphasize lesson design and how to hold classroom mathematical conversations
- Examples
Curriculum & Assessment (5th semester)

- Teaching Gap (Stigler & Hiebert, 1999)
- International comparisons of curricula (Schmidt and others)
- Examination of other countries’ elementary and secondary textbooks (Japan, Singapore, Finland)
  - Emphasize development of mathematical meanings and ways of thinking over time
- Learning goals and forms of assessment
  - Didactic triad
  - Formative and summative assessments; examples from US and other countries
Development of Mathematical Thinking (7th semester)

- Research on additive and multiplicative reasoning, and their development

- Design and conduct a teaching experiment with one school student on a foundational mathematical idea

- Analyze and report findings
Thank You
pat@pat-thompson.net