1. Randomness in Diophantine approximation

Joint work with Ghosh.

Counting lattice points in a region: Theorem (Schmidt) Ω_T is an increasing family of Borel sets in $\mathbb{R}^d$ of finite measure, the volume $|\Omega_T| \to \infty$ as $T \to \infty$. Then, for a.e. unimodular lattice in $\mathbb{R}^d$, $\#(\Lambda \cap \Omega_T) \sim |\Omega_T| + \begin{cases} O_\Lambda(|\Omega_T|^{\frac{1}{2}}(\log|\Omega_T|)^{\frac{3}{2}+\epsilon} & d \geq 3 \\ O_\Lambda(|\Omega_T|^{\frac{1}{2}}(\log|\Omega_T|)^{\frac{5}{2}+\epsilon} & d = 2 \end{cases}$.

The naive heuristics of the above result is as follows: $\Omega_T = \amalg \Omega_i$, $|\Omega_i| \sim 1$, then the expected number of lattice points in each Ω_i is about $|\Omega_i|$. If they behaves as if they are independent, there should be a central limit theorem-like result.

Example: When Ω_T is chosen to be rational ellipses of size T, Landraw and Walfize showed that the error term for $\#(\mathbb{Z}^d \cap \Omega_T)$ is $O(T^{d-2})$, which is the best possible result. The error term comes from the $O(T^{d-1})$ unit cubes that touch the boundary of the ellipse, which behave with some kind of "independence".

Conjecture (Gotze) for a.e. unimodular lattices the above is true for error $O(T^{\frac{d-1}{2}+\epsilon}$.

Relationship with Diophantine approximation:

Let $A : \mathbb{R}^{m+n} \to \mathbb{R}^n$, $\Omega_T = \{x | ||Ax|| \leq ||x||^{-\frac{m}{n}}, 1 \leq ||x|| \leq T\}$.

Theorem (G-G) If $m + n \geq 3$, then $\exists \sigma > 0$, (1) the volume of unimodular lattices in $\mathbb{R}^{n+m}$ such that $\frac{\#(\Lambda\cap\Omega_T)-|\Omega_T|}{|\Omega_T^{1/2}|} \in (a, b)$ converges to the probability of a random variable $\sim \mathcal{N}(0, \sigma^2)$ lies in (a, b), as $T \to \infty$. (2) $\limsup_{T\to\infty} \frac{\#(\Lambda\cap\Omega_T)-|\Omega_T|}{|\Omega_T|^{\frac{1}{2}}(\log\log|\Omega_T|)^{\frac{1}{2}}} = \sqrt{2}\sigma$.

When $m + n = 2$, replace $\frac{\#(\Lambda\cap\Omega_T)-|\Omega_T|}{|\Omega_T^{1/2}|}$ with $\frac{\#(\Lambda\cap\Omega_T)-\zeta(2)^{-1}|\Omega_T|}{|\Omega_T^{1/2}|}$.

Idea of the proof: decompose $\Omega_T = \{(u, v) \in \mathbb{R}^{n+m} | ||u|| \leq c||v||^{-m/n}, 1 \leq ||v|| \leq T\}$ into regions where $||v||$ is between 1 and 2, 2 and 4, 4 and 8 etc. Each of these cylindrical region is obtained by a linear transformation $a = diag(2^{\frac{m}{n}} I_n, \frac{1}{2} I_m)$ from the first one, denoted as Ω_0.

Now use Siegel transform: $\#(\Lambda \cap \Omega_{2^k}) = \sum_i^{k-1} \sum_{x\in\Lambda} \chi_{\Omega_0}(a^i x) = \sum_i \hat{\chi_{\Omega_0}}(\Lambda)$.

Theorem: $T : X \to X$ partially hyperbolic diffeomorphism with some kind of mixing, f a Hölder function on X with compact support, and is not a coboundary ($f \neq gT - g$), then the average of $f(T^j X)$ satisfies central limit theorem.

However, the Siegel transform is not smooth nor bounded. However, it is in L^2 when dimension ≥ 3.

The proof is similar to central limit theorem:

Theorem (Goedin) $T : (X, \mu) \to (X, \mu)$, invertable, measure-preserving, and there is a filtration of σ-algebras $\mathcal{C}_n \subset \mathcal{C}_{n+1}$, related by T^{-1}, $f \in L^2$ with 0 average, such that (1) $\sigma^2 = \sum_{i\in\mathbb{Z}}(T^i f, f) \in (0, \infty)$, and (2) $\sum_{i>0} ||E(f|\mathcal{C}_n) - f||_2 < \infty$, $\sum_{i>0} ||E(f|\mathcal{C}_n)||_2 < \infty$, then $\frac{1}{\sqrt{n}}\sum_{i=1}^n f(T^i x) \sim \mathcal{N}(0, \sigma)$.

To construct these $\mathcal{C}_n$, let Q be a partition of X, $P(x) = Q(x) \cap W_a^u(x)$, and $\mathcal{C}_n(x) = \cap_{i=-n}^{\infty} a^i P(a^{-i}x)$.

To deal with some badly behaved parts, fix $\rho < 1$, let $X(\eta) = \{x \in X | B^u_{\eta\rho^k}(a^{-k}x) \subset \mathcal{C}_0(a^{-k}x), \forall k \geq 0\}$. $\mu(X - X(\eta)) << \eta^c$, hence, $E(f|\mathcal{C}_n) = \frac{1}{m^u(a^{-n}\mathcal{C}_0(a^n x))}\int_{\mathcal{C}_0(a^n x)} f(y) dm^u(y)$.