COUNTING AND DYNAMICS IN SL_2

MSRI

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MOTIVATION: 2 STRIKING CONJECTURES

CONTINUED FRACTIONS:

IF $x \in (0,1)$ $\quad x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_k + \cdots}}}$ $\quad *$

WRITE $x = [a_1, a_2, \ldots]$.

FINITE $\Leftrightarrow$ x RATIONAL

$\mathcal{A} \subseteq \mathbb{N}$ FINITE ALPHABET

$C_{\mathcal{A}}$ $\quad \infty$ CONTINUED FRACTIONS: $a_i \in \mathcal{A}$ $\forall i$
AS IN $*$

$R_{\mathcal{A}}$ $\quad$ APPROXIMANTS $\quad \{ [a_1, \ldots, a_k] : \underset{\text{each } i}{a_i \in \mathcal{A}} \}$.

SOMETHING IN $C_{\mathcal{A}}$ FOR SOME FIN. $\mathcal{A}$

CALLED <u>ABSOLUTELY DIOPHANTINE</u>

D_A DENOMINATORS OF R_A.

LET $[A] = \{1, \ldots, A\}$.

CONJECTURE (ZAREMBA) $\exists A$ SUCH THAT $D_{[A]} = \mathbb{N}$.

THEOREM (BOURGAIN - KONTOROVICH).

$\exists A$ SUCH THAT

$$\left| D_{[A]} \cap \{1, 2, \ldots, N\} \right| = N + O\left(N^{1 - \frac{c}{\log\log N}}\right)$$

CONJECTURE (McMULLEN - 'ARITHMETIC CHAOS').

$\exists\ A \geq 2$ SUCH THAT $\forall$ REAL QUADRATIC FIELDS K

$$\left| \{ [\overline{a_0, \dots, a_\ell}] \in K \ : \ a_j \in [A] \} \right|$$

GROWS EXPONENTIALLY IN ℓ.

CURRENTLY OPEN: IS THIS SET NON-EMPTY FOR ALL K?

LATTICE POINT COUNT FOR THIN SEMIGROUPS

$G_{\mathcal{A}}$ = ALL POSSIBLE PRODUCTS OF $g_a = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$
$a \in \mathcal{A}$.

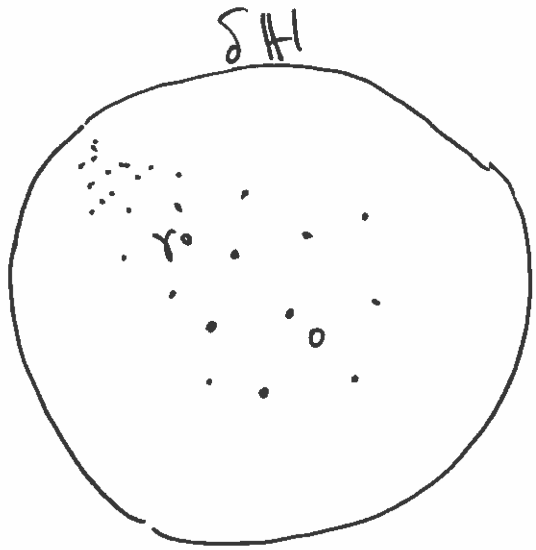
THEN $\mathcal{D}_{\mathcal{A}} = \left\langle G_{\mathcal{A}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

$\Gamma = \Gamma_{\mathcal{A}} = \langle g_{a_1} g_{a_2} : a_1, a_2 \in \mathcal{A} \rangle_{\text{SEMIGROUP}}$

Γ IS A <u>THIN</u> SEMIGROUP IN $SL_2(\mathbb{Z})$.

THERE IS NO SURFACE $\sim \Gamma$

<u>THIN</u> POINCARÉ BALL PICK $o \in \mathbb{H}$



Γo ACCUMULATES ON $\delta \mathbb{H}$

$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$ BY MÖBIUS.

$L(\Gamma)$ = SET OF ACCUMULATION POINTS OF Γo

$$\delta = \delta(\Gamma_A) = \dim_{\text{Hausdorff}}(L(\Gamma))$$

<u>CONNECTION</u>

$$L(\Gamma_A) = C_A$$

HAUSDORFF DIMENSIONS.

THM (HENSLEY '92).

$$\delta_{[A]} = 1 - \frac{6}{\pi^2 A} - \frac{72 \log A}{\pi^4 A^2} + O\left(\frac{1}{A^2}\right).$$

FOLLOWS FROM DYNAMICAL SETUP

THAT. IF $|\mathcal{A}| < \infty$.

$\delta < 1$.

Γ THIN

ALSO FROM DYNAMICAL SETUP.

Γ FREE SEMIGROUP.

NORM ON MATRICES $\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = (a^2 + b^2 + c^2 + d^2)^{1/2}$.

THEOREM (M, OH, WINTER)

$\exists\ q_0$ SUCH THAT WHEN q SQUAREFREE AND $(q, q_0) = 1$.

$\epsilon > 0$ $\quad \forall\ \xi \in SL_2(\mathbb{Z}/q\mathbb{Z}) = \Gamma_q$

$$\sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| \leq R \\ \gamma \equiv \xi \bmod q}} 1 = \frac{C_\Gamma R^{2\delta}}{|\Gamma_q|} + O(q^C R^{2\delta - \epsilon}).$$

(A) WHEN $\Gamma = SL_2(\mathbb{Z})$, RESULT

FOLLOWS FROM BOUNDS ON EIGENVALUES

OF AUTOMORPHIC FORMS

E.G. SELBERG'S 3/16 THEOREM.

(B) OUR RESULT ALLOWS FOR SECTOR

ESTIMATES.

(C) BOURGAIN - GAMBURD - SARNAK GET

RESULT WITH $R^{\left(2\delta - \frac{c}{\log\log R}\right)}$ ERROR

WHEN Γ THIN (E.G.) SUBGROUP OF $SL_2(\mathbb{Z})$

WITH $\delta(\Gamma) \leq 1/2$

AND UNIFORM ERROR WHEN $\delta > 1/2$

ii) OH AND WINTER EXTENDED BOURGAIN - GAMBURD - SARNAK RESULT TO GET UNIFORM ERROR FOR ANY F.G THIN SUBGROUP OF $SL_2(\mathbb{Z})$

- BREAKTHROUGH TO LEVEL OF DIFFICULTY REQUIRED FOR OUR MAIN THEOREM.

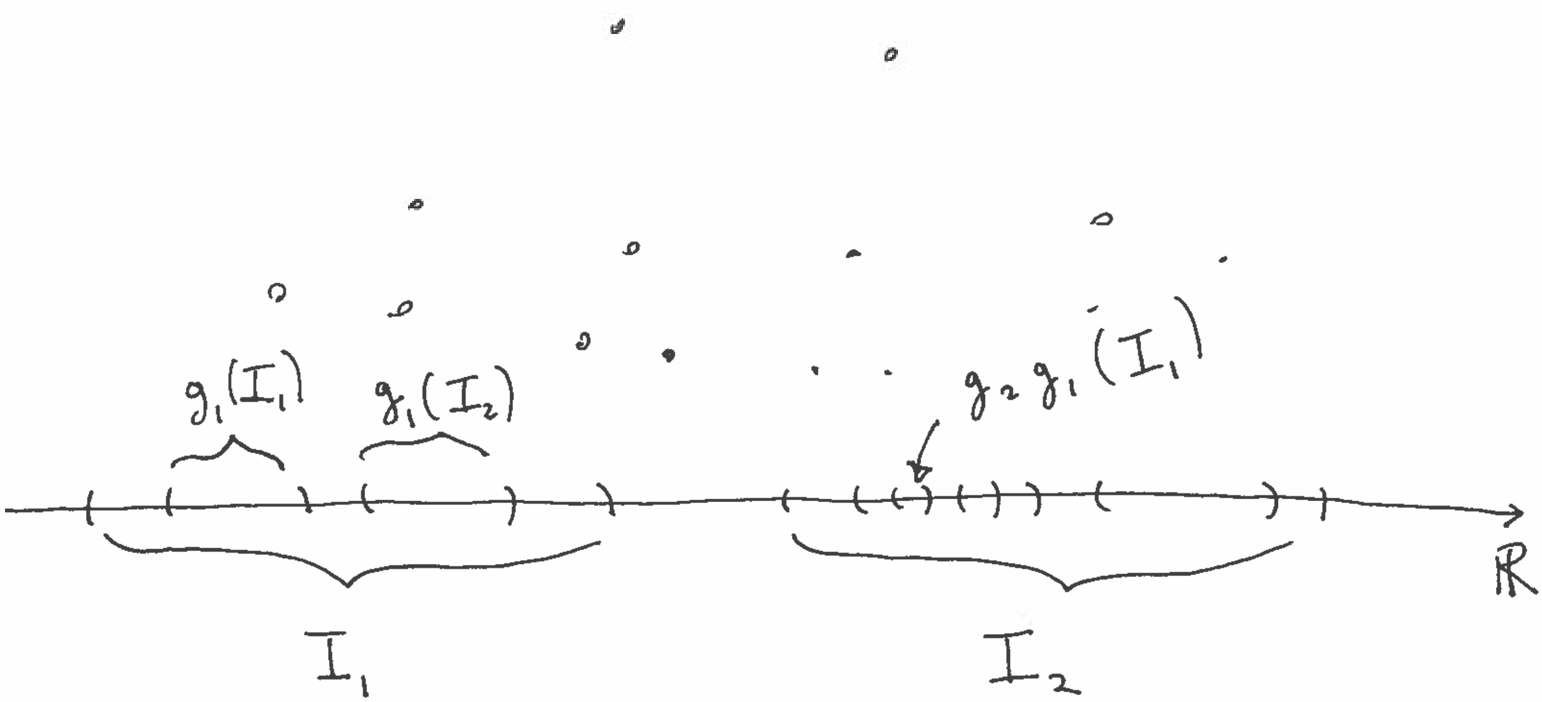
(E) LATTICE POINT COUNT OF THEOREM FEATURES IN BOURGAIN - KONTOROVICH WORK ON ZAREMBA'S CONJECTURE (THEY USE $R^{2\delta - c/\log\log R}$ ERROR) IN SUCH A WAY THAT SUGGESTS ONE MAY MAKE FURTHER PROGRESS WITH UNIFORM ERROR

SET UP.

FREE SEMIGROUP

$$\langle g_1, \dots, g_k \rangle_{\text{SEMIGROUP}}$$

$$\cdot\ 0 \in \mathbb{H}$$



FIND INTERVALS $\{I_i\}$

SUCH THAT $g_i(I_j) \subseteq I_i \ \forall j$

LIMIT SET $L(\Gamma) \sim$ INFINITE INTERSECTIONS OF NESTED INTERVALS

DISTORTION

\# ELEMENTS IN Γ OF WORD LENGTH $\leq L$ $= K^L + O(K^{L-1})$.

WE WANT TO COUNT. $\# \{ \Gamma_0 \cap B_{\mathbb{H}}(0, T) \}$

(AND $\gamma \equiv \xi \bmod q$).

W/O CONGRUENCE CONDITION, CORRECT MAIN TERM IS $e^{\delta T}$

MOREOVER THERE IS A FUNCTION

$$\tau : L(\Gamma) \longrightarrow \mathbb{R}$$

CALLED DISTORTION FUNCTION

EXTENDS TO REAL ANALYTIC ON $I = \bigcup I_i$

FUNCTION

THE 'SHIFT'

HAVE MAP $T : I \to \mathbb{R}$

DEFINED BY $T|_{I_i} = g_i^{-1}$.

DISTORTION FUNCTION: $\tau(x) = \log|T'(x)|$

T EXPANDING: $|T'(z)| > 1$
$z \in I$

$\Rightarrow \quad \tau > c > 0$ ON I.

HAVE MARKOV PROPERTY

: $T(I_j) \cap I_i \neq \phi$
$\Rightarrow T(I_j) \supseteq I_i$

LATTICE POINT COUNT

GOOD: HAVE TREE STRUCTURE

BAD: HAVE WARPED METRIC

FOLLOWING LALLEY, CONSIDER

$$N^*(a, \gamma_0) = \sum_{\gamma \in \Gamma \cup \{1\}} \mathbb{1}\{ d(o, \gamma\gamma_0 o) - d(o, \gamma_0 o) \leq a \}.$$

COUNTING PARAMETER

PARAMETER FOR RECURSION.

$\gamma_0 = id \longrightarrow$ WHAT WE WANT (AFTER CONGRUENCE CONDITION ADDED).

RENEWAL :

RELATES $N^*(a, \gamma_0)$

TO SUM OVER $N^*(a', \gamma_0')$

WHERE $\gamma_0' = g_i \gamma_0$ FOR SOME g_i

MOVE TO BOUNDARY

<u>IDEA</u>: ALL ACTION OF LATTICE POINT COUNT TAKES PLACE CLOSE TO $L(\Gamma) \subset \partial \mathbb{H}$

$$\gamma, \gamma_0 \longrightarrow \text{POINTS IN } I.$$

$$N^* \longrightarrow N(a, x)$$

$$= \sum_{n=0}^{\infty} \sum_{y : T^n y = x} \mathbb{1}\{\tau^n(y) \le a\}$$

WHERE $\tau^n(y) = \sum_{i=0}^{n-1} \tau(T^i y)$.

'DISTORTION ALONG TRAJECTORY'.

... TO TRANSFER OPERATORS.

i) TRANSLATE RENEWAL TO ANALOG FOR $N(a, x)$

ii) TAKE LAPLACE TRANSFORM IN a VARIABLE

GET.

$$s \hat{N}(s, x) = (1 - \mathcal{L}_s)^{-1}[1](x).$$

$$\mathcal{L}_s : C^1(I) \longrightarrow C^1(I)$$

$$\mathcal{L}_s[f](x) = \sum_{y : Ty = x} e^{-s\tau(y)} f(y).$$

IN CONGRUENCE LATTICE POINT COUNT.

EXTEND $\mathcal{L}_s$ TO TWISTED VERSION

$\mathcal{L}_{s,q}$ ACTING ON $\mathbb{C}\Gamma_q$, $\Gamma_q = SL_2(\mathbb{Z}/q\mathbb{Z})$

GROUP-ALGEBRA VALUED

FUNCTIONS. $f \in C^1(I, \mathbb{C}^{\Gamma_q})$.

$$\mathcal{L}_{s,q}[f] = \sum_{y : Ty = x} e^{-s\tau(y)} \underbrace{c_q(y) f(y)}.$$

$c_q(y) = g_i \bmod q$ WHEN.

y IN I_i

ACTION BY REGULAR REPRESENTATION.

NEED STRONG SPECTRAL BOUNDS FOR

$\mathcal{L}_{s,q}$ IN TWO ASPECTS.

Im(s) large, $s = a + ib$

$$\mathcal{L}_{s,q}[f](x) \approx \sum_{y:\, T(y)=x} \boxed{e^{-ib\tau(y)}} f(y).$$

REQUIRES REAL DYNAMICS.

q LARGE

$$\mathcal{L}_{s,q}[f](x) \approx \sum_{y:\, T(y)=x} \boxed{c_q(y)} f(y)$$

REQUIRES MOD q DYNAMICS.

REAL DYNAMICS

EXTEND WORK OF DOLGOPYAT TO VECTOR VALUED FUNCTIONS - FIRST DONE BY OH AND WINTER

PROVE L_s CONTRACTING FOR $Re(s) \sim \delta$ AND $Im(s)$ LARGE

NORMALIZED VERSION OF $\mathcal{L}_s$

BY CONSIDERING

$$\underbrace{L_s^N L_s^N \cdots L_s^N}_{n/N}(L_s^N f)$$

'REPLACEMENT'

$$L_s^N \cdots (L_s^N f_1)$$

'REPLACEMENT'.

$$L_s^N f_2$$

$$\cdots$$

$$f_{n/N}$$

GET SAVING EACH TIME.

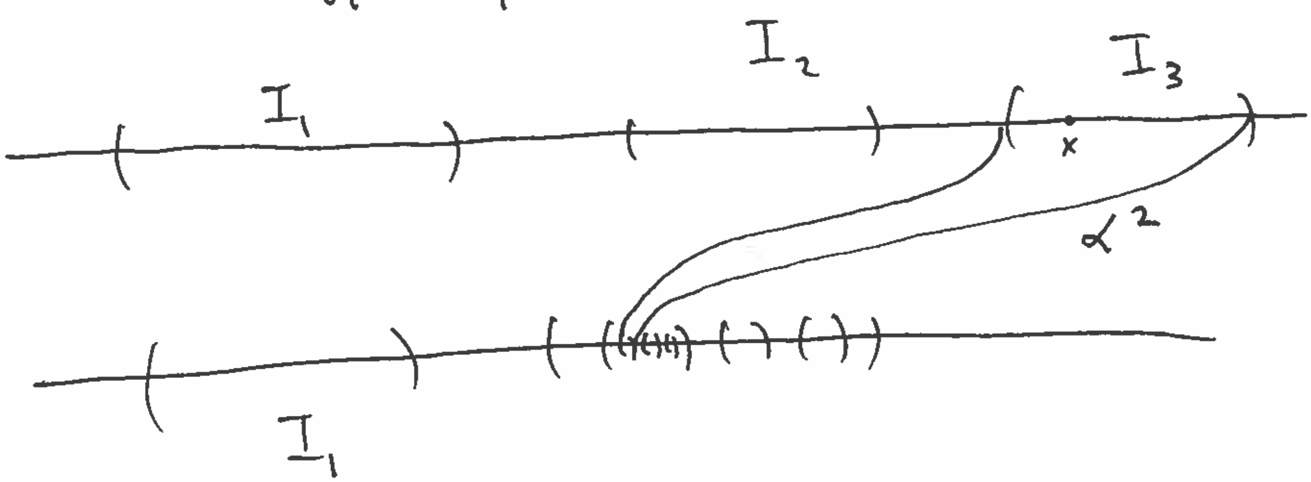
REPLACEMENT CALLED DOLGOPYAT OPERATOR

KEY POINTS

- KEEP TRACK OF REGULARITY USING BIRKHOFF CONES

- $$L_s^N[f](x) = \sum_{\alpha^N} e^{-s\tau^N(\alpha^N(x))} f(\alpha^N(x)) \quad (*)$$

α^N LOCALLY DEFINED INVERSES OF T^N



SINCE $\mathrm{Im}(s)$ LARGE, END UP WITH OSCILLATORY INTEGRALS IN EVALUATING L^2 NORM OF $L_s^N[f]$.

$s = a + ib$

$$\int_I |L_s^N[f]|^2(x)\, d\nu(x)$$

$$(*) = \sum_{\alpha_1^N, \alpha_2^N} \int_I e^{-a(\cdot\cdot)} e^{-ib\left(\tau^N(\alpha_1^N x) - \tau^N(\alpha_2^N x)\right)} \times f(\alpha_1^N x)\overline{f}(\alpha_2^N x)\, d\nu(x).$$

GETTING SAVING FROM ONE SUMMAND ENOUGH.

τ HAS 'NON LOCAL INTEGRABILITY' PROPERTY.

$\Rightarrow$ SOURCE OF α_1^N, α_2^N SUCH THAT

$$\left| \frac{d}{dx}\left(\tau^N \circ \alpha_1^N - \tau^N \circ \alpha_2^N \right) \right| > c$$

USE NON STATIONARY PHASE IN $(*)$

USING $|b|$ LARGE.

PROBLEMS TO OVERCOME $\left(\text{GIVEN } NLI_{\tau} \text{ FOR} \right)$.

ν IS SUPPORTED ON $L(\Gamma)$,

A CANTOR SET.

'CALCULUS ON FRACTALS'.

FIND OSCILLATIONS. 'BY HAND'

USING ν FEDERER AND

CUT UP I INTO 'TRIADIC PARTITION'

OF $L(\Gamma)$. ON SCALE $\frac{1}{|b|}$.

KEY FEATURE OF τ IS NLI.

SOURCE OF THIS NLI FOR τ.

'ANOSOV ALTERNATIVE' GIVES DICHOTOMY

THAT ALONG WITH REAL ANALYTICITY

OF τ IMPLY IT IS ENOUGH

TO SHOW THERE ARE

$$\gamma_1, \gamma_2 \text{ in } \Gamma$$

SUCH THAT $|\mathrm{tr}(\gamma_1\gamma_2\gamma_1\gamma_2)| \neq |\mathrm{tr}(\gamma_1^2\gamma_2^2)|$.

(USE RELATIONSHIP OF τ WITH LENGTH OF PERIODIC ORBITS AND TRACES).

~~BUT~~ USE

$$\mathrm{tr}(\gamma_1\gamma_2\gamma_1\gamma_2) - \mathrm{tr}(\gamma_1^2\gamma_2^2)$$

$$= \mathrm{tr}([\gamma_1, \gamma_2]) - 2.$$

A BRIEF ACCOUNT OF MOD p DYNAMICS

(SQUAREFREE IS SIMILAR).

NEED TO INJECT MEASURE FLATTENING RESULTS OF BOURGAIN-GAMBURD, BOURGAIN-GAMBURD-SARNAK.

ORIGINALLY MOTIVATED BY WORK OF BOURGAIN-GAMBURD ESTABLISHING EXPANSION OF CAYLEY GRAPHS IN $SL_2(\mathbb{F}_p)$.

REST ON TRIPLING ESTIMATES OF HELFGOTT IN $SL_2(\mathbb{F}_p)$

IN TURN RESTS ON SUM-PRODUCT THEOREM OF BOURGAIN-KATZ-TAO.

SOURCE OF DYNAMICS

THE END.

THANK YOU.