Introduction to Teichmüller Spaces

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1. Riemann Surfaces

Definition 1.1. A *conformal structure* is an atlas on a manifold such that the differentials of the transition maps lie in $\mathbb{R}_+ \times SO(n)$.

Definition 1.2. A *Riemann surface* is a 2-dimensional manifold together with a conformal structure; or, equivalently, a 1-dimensional complex manifold.

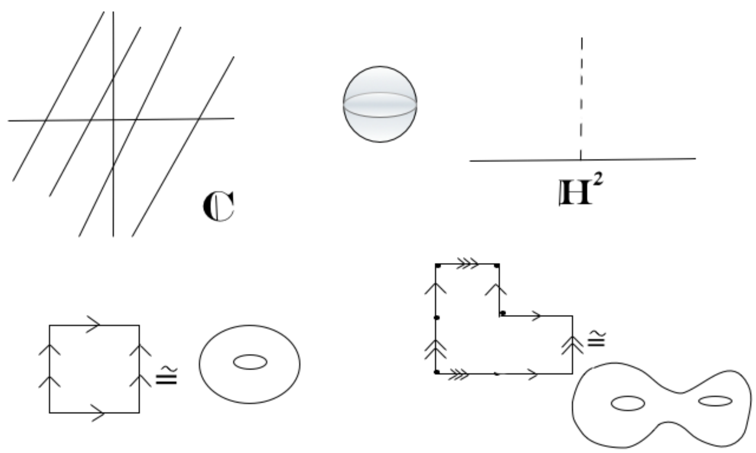

Figure 1: Examples of Riemann Surfaces

1.1 Riemann's Goal

Riemann's goal was to classify all Riemann surfaces up to isomorphism; i.e. up to biholomorphic maps.

There are two types of invariants:

- *discrete* invariants, which arise from topology (for example, genus)
- *continuous* invariants (called *moduli*), which come from *deforming* a conformal structure.

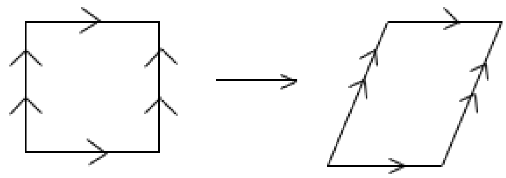

Figure 2: Conformal Deformation

1.2 Riemann's Idea

Riemann's idea was that the space of all closed Riemann surfaces up to isomorphism is a "manifold", a geometric and topological object:

$$\begin{aligned} M &= \{\text{closed Riemann surfaces}\}/\sim \\ &= \bigcup_{g\geq 0} M_g, \end{aligned}$$

where $M_g = \{\text{genus g Riemann surfaces}\}/\sim$ is a connected component of M. Now the goal is to understand the topology and geometry of each M_g.

2. Uniformization

We will now investigate why genus is the only discrete invariant. Given a Riemann surface X_g, its conformal structure lifts to its universal cover, $\tilde{X}_g$. Uniformization Theorem says:

$$\tilde{X}_g := \begin{cases} \hat{\mathbb{C}} & \text{if } g = 0 \\ \mathbb{C} & \text{if } g = 1 \\ \mathbb{H}^2 & \text{if } g > 2 \end{cases}$$

Remarks.

i. Each of $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H}^2$ has a distinct natural conformal structure.

ii. For g=0, $X_g \cong \hat{\mathbb{C}}$ so $M_0 = \{\hat{\mathbb{C}}\}$.

iii. Each of $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H}^2$ admits a Riemannian metric of constant curvature, which is compatible with its natural conformal structure.

	$\hat{\mathbb{C}}$	$\mathbb{C}$	$\mathbb{H}^2$
κ	1	0	-1

So X_g admits a metric of constant κ, and we can identify

$$M_g = \{\text{genus g Riemann surfaces with constant curvature }\}/\text{isometry}$$

(For g=1, we need to normalize area as well.)

3. Teichmüller Space

We fix a topological surface S of genus g.

DEFINITION 3.1. A *marked Riemann surface* (X, f) is a Riemann surface X together with a homemorphism $f : S \to X$. Two marked surfaces $(X, f) \sim (Y, g)$ are equivalent if $gf^{-1} : X \to Y$ is isotopic to an isomorphism.

DEFINITION 3.2. We define the Teichmüler Space

$$T_g = \{(X, f)\}/\sim$$

For $g \geq 2$, T_g is also the set of marked hyperbolic surface (X, f), where the equivalent relation is given by isotopy to an isometry.

There is a natural forgetful map $T_g \to M_g$ by sending $(X, f) \mapsto X$. We note that (X, f) and (X, g) are equivalent in M_g if and only if exists an element $h \in \text{Homeo}^+(S)$ such that $f = gh^{-1}$, where h well-defined up to isotopy. This introduces:

DEFINITION 3.3. The *mapping class group* is

$$\Gamma_g = \text{Homeo}^+(S)/\text{Homeo}_0(S),$$

where $\text{Home}_0(S)$ is the connected component of the identity.

We define an action of $\Gamma_g \curvearrowright T_g$ by $(X, f) \mapsto (X, fh^{-1})$. By the above discussion, $T_g/\Gamma_g = M_g$.

5. Topology on T_g

Teichmüller space T_g is naturally a manifold homeomorphic to $\mathbb{R}^{6g-g}$, and Γ_g acts properly discontinuously on T_g. Thus, M_g is an orbifold with $\pi_1^{\text{orb}}(M_g) = \Gamma_g$.

We are able to see the topology in two ways:

By Representation theory:

$$T_g \hookrightarrow \text{Hom}(\pi_1(S), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}) = \text{char}_2(\pi_1(S)),$$

where the image of T_g is the open subset of discrete and faithful representations. A simple counting argument shows

$$\dim(\Gamma_g) = \dim \text{char}_2(G) = (2g-1) * 3 - 3 = 6g - 6.$$

By Fenchel-Nielson Coordinates:

Example 5.1. Dehn Twist: We define an element $D_\alpha \in \Gamma_g$, where α is a simple closed curve on S.

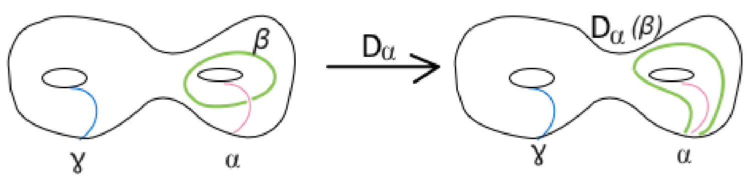

Figure 3: Dehn Twist

Example 5.2. Fenchel-Nielson coordinates on $T_{1,1}$ (The Teichmüller space of the once-punctured torus):

Given the once-punctured torus S. Fix α, β on S, α will be a *pants decomposition* of S and β a *seam*. Let $(X, f) \in T_{1,1}$. As shown in Figure 4, then the map f identifies α with a curve (also called) α in X. Let $\ell = \ell_X(\alpha)$ be the length of the unique geodesic in X in the homotopy class of α.

As seen on the right side of the figure, in hyperbolic geometry, there exists a unique arc γ that intersects α perpendicularly on both sides. Let ω be the arc in α between the foots of the of ω. Now let $\beta' = \gamma \cup \omega$. This is a closed curve which differs from the image of β in X by some power of Dehn twist along α, i.e. $\beta' = D_\alpha^n(\beta)$.

We define

$$\tau = n\ell + \ell_x(\omega)$$

Definition 5.3 (FN Coordinates). The Fenchel-Nielsen coordinates relative to the curves (α, β) is

$$T_{1,1} \to \mathbb{R}_+ \times \mathbb{R}, X \mapsto (\ell, \tau)$$

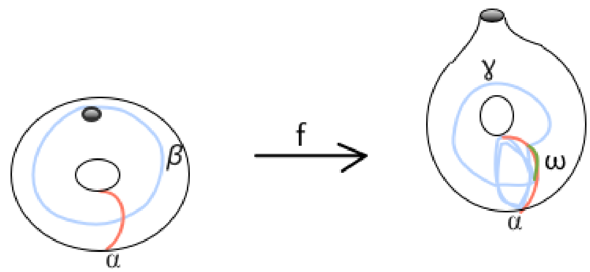

Figure 4: FN on $T_{1,1}$

In general (for higher-dimensional cases), we need to fix a pants decomposition $\Sigma = \{\alpha_1, ..., \alpha_{3g-3}\}$ on S and a set of $3g-3$ seams. Then the FN coordinates relative to Σ is

$$T_g \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

$$X \mapsto (\ell_1, ..., \ell_{3g-3}, \tau_1, ..., \tau_{3g-3})$$

6. Teichmüller Metric

(Or how to compare conformal structures)

If two points in Teichmüller space $(X, f) \neq (Y, g)$, then $gf^{-1} : X \to Y$ is not homotopic to a conformal map. Our goal is to quantify how far gf^{-1} is from being conformal.

Let $h : X \to Y$ be an *orientation-preserving* diffeomorphism. For $p \in X$,, we have

$$(dh)_p : T_pX \to T_{f(p)}Y$$

$(dh)_p$ is $\mathbb{R}$-linear, but not necessarily $\mathbb{C}$-linear. There is a decomposition

$$(dh)_p = R \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} S,$$

where R and S are rotations, and $a, b > 0$.

Definition 6.1. The *dilatation* at p as

$$K_p = \frac{\max\{a,b\}}{\min\{a,b\}} \geq 1$$

Definition 6.2. The *dilatation* of h is

$$K_h = \sup_p K_p \geq 1$$

We have:
(i) $(dh)_p$ is $\mathbb{C}$-linear iff $a = b$ iff $K_p = 1$
(ii) h is conformal iff $K_h = 1$.

Definition 6.4. h is a quasi-conformal map if $K_h < \infty$. This holds automatically if X is compact.

Definition 6.3 (Teichmüller Distance). The define the *Teichmüller Distance* is

$$d_T((X,f),(Y,g)) = \frac{1}{2}\log \inf_{h\sim gf^{-1}} K_h$$

where $\inf_{h\sim gf^{-1}} K_h$ is the smallest dilatation of a quasi-conformal map preserving the marking.

Lemma. d_T is a metric.

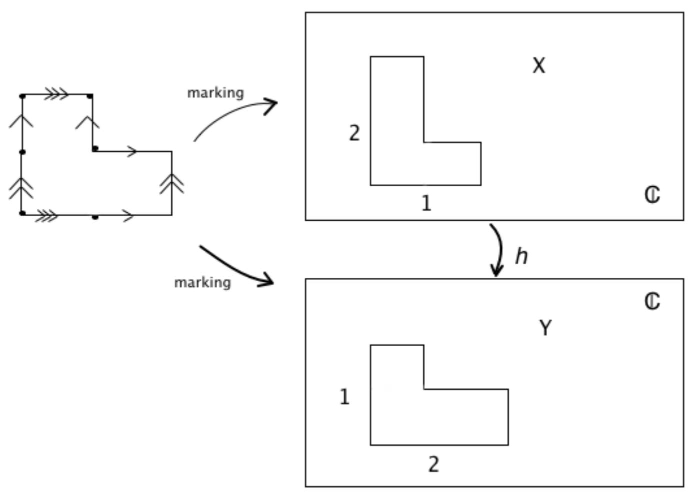

Figure 5: Ex. of extremal map h

Example.
Consider

$$h = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

We see $K_h = 4$. h turns out to be the unique extremal map. This means that any map $h' \sim h$ has bigger dilatation, $K_{h'} > K_h$. Hence $d_T(X, Y) = \frac{\log(4)}{2}$.

Definition 6.5 (Quadratic Differential). A *quadratic differential* on $X \in T_g$, is $q : TX \to \mathbb{C}$. Locally, q has the form $q = q(z)dz^2$ where $q(z)$ is holomorphic.

Remark. q has $4g - 4$ zeroes counted with multiplicity.

Definition 6.6. If p not a zero of q, $q(0) \neq 0$ in local coordinates, then we can take a branch of $\sqrt{q(z)}$ and integrate to obtain a *natural coordinates* ω for q:

$$\omega = \int \sqrt{q(z)}dz, \quad q = d\omega^2$$

The transition of natural coordinates (or the change of charts between natural coordinates) includes translations and possible sign flip, since $d\omega^2 = (d\omega')^2$ so $\omega' = \pm\omega + c$.

So ω defines a (singular) flat Euclidean metric $|d\omega|^2$ on X (singularities come from the zeros of q). Conversely, a collection of natural coordinates determines a quadratic differential.

Example.
If we take X from the previous example, then let $q = dz^2$.

Let $QD = \{$quadratic differentials on X$\}$. By Riemann-Roch, QD is a complex vector space of $\dim_{\mathbb{C}} = 3g - 3$. Also, $QD(X) = T^*_x(T_g) = T^*_x(M_g)$.

Definition 6.7. We define an L^1 norm on $QD(X)$. Let $q = q(z)dz^2$. Let

$$||q||_1 = \int |q(z)|dzd\bar{z}$$

This is just the area of X in the (singular) flat metric.

Definition 6.8. For a point $X \in T_g$ and $q \in QD(X)$, denote the open unit ball by $QD^1(X) = \{||q|| < 1\}$.

Definition 6.9 (Teichmüller Map).

For $X \in T_g$ and $q \in QD^1(X)$, let

$$K = \frac{1 + ||q||}{1 - ||q||} \geq 1.$$

Set $\omega = u + iv$ to be a natural coordinate for q, and define *a new natural coordinate* by $\omega' = \sqrt{K}u + i\frac{1}{\sqrt{K}}v$. This new coordinate ω' determines a surface $Y_q \in T_g$ and a canonical map $X \xrightarrow{h_q} Y_q$, called a Teichmüller map.

THEOREM 6.10. We have
(i) h_g is the unique extremal map in its homotopy class.
(ii) $QD^1(X) \to T_g$ such that $q \mapsto Y_q$ is a homeomorphism.

CONSEQUENCES.
(i) d_T is complete.
(ii) $t \mapsto e^{\frac{t}{2}}u + ie^{\frac{-t}{2}}v$ defines a bi-infinite geodesic line in this metric.
(iii) Any $X, Y \in T_g$ is connected by one and only one segment of such a line.

REMARKS.
(i) $(T, d_T) \cong (\mathbb{H}^2, \text{hyperbolic metric})$ but for $g \geq 2$, (T_g, d_T) is not hyperbolic in any sense. (Masur, Masur-Wolf, Minsky)
(ii) Geodesic rays do not always converge in the Thurston boundary. (Lenzhen)
(iii) (Masur-Minsky, Rafi) gave a combinatorial descriptions of Teichmüller geodesics.

7. Weil-Petersson Metric
(or L^2-norm on $QD(X)$)

A point $X \in T_g$ is a hyperbolic surface. Write the hyperbolic metric in local coordinates as $ds^2 = \rho(z)|dz|^2$. For $q_1, q_2 \in T_g$, define a Hermetian inner prodcut on $QD(X)$ by

$$h(q_1, q_2) = \int_X \frac{q_1(z)\overline{q_2(z)}}{\rho(z)} dz d\bar{z}$$

REMARKS.
(T_g, h) is a Kähler manifold, that is T_g has three natural structures that are all compatible with each other:

- a complex structure
- a Riemannian structure, the associated Riemannian metric – called the Weil-Peterssoon metric – is $g_{\omega p} = \text{Real}(h)$
- and a symplectic structure, the associated WP–symplectic form (i.e. a closed $(1,1)$ form) is $\omega = -\text{Im}(h)$.

THEOREM 7.1 (WALPERT'S FORMULA).
Choose a set of FN coordinates on T_g

$$\Phi : T_g \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

$$X \mapsto (\ell_1, ..., \ell_{3g-3}, \tau_1, ..., \tau_{3g-3})$$

Then the WP sympletic form is

$$\omega = \frac{1}{2} \sum_{i=1}^{3g-3} d\ell_i \wedge d\tau_i$$

Example.
For $T_{1,1}$, its natural complex structure is $\mathbb{H}^2$. For y large, $\tau \sim \frac{x}{y}, \ell \sim \frac{x}{y}$, therefore

$$\omega = d\ell \wedge d\tau \sim \frac{1}{y^3}(dx \wedge dy),$$

thus

$$g_{wp} \sim \frac{1}{y^3}(dx^2 + dy^2)$$

when y is large.

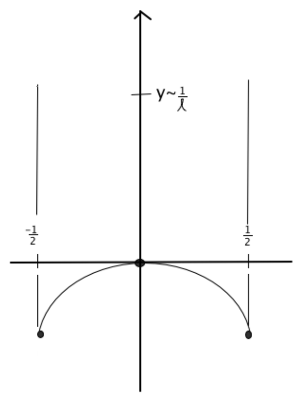

Figure 6: $T_{1,1}$ Ex. of Walpert's Formula

We that that the arc length of the imaginary axis $\int \frac{1}{y^{3}2}|dz| < \infty$. This implies that g_{wp} is incomplete.

Also, $\kappa_{wp} \sim -y$ for y large, so g_{wp} has negative Gaussian curvature with sup $\kappa = -\infty$. But κ_{wp} is bounded away from 0.

Remarks.
(i) In general, the WP metric is always incomplete.

(ii) It always has negative sectional curvature, but for $\dim_{\mathbb{C}}(T_g) > 2$, $\sup \kappa_{wp} = 0$ and $\inf \kappa_{wp} = -\infty$ (Huang).
(ii) (Brock) showed (T_g, g_{wp}) is quasi-isomorphic to a pants graph.

8. Thurston Metric

(or how to compare hyperbolic structures)

DEFINITION 8.1.
A map $h : X \to Y$ is a K_h-Lipschitz map

$$d(h(x), h(y)) \leq K_h d(x, y)$$

DEFINITION 8.2. For $X, Y \in T_g$, define

$$L(X, Y) = \inf_{h \sim gf^{-1}} K_h$$

where h is a Lipschitz homeomorphism.

LEMMA (THURSTON). $L(X, Y) \geq 1$ and is not necessarily symmetric.

DEFINITION 8.3 (THURSTON DISTANCE). The *Thurston distance* is $d_L(X, Y) = \log L(X, Y)$ which by the preceding lemma is an asymmetric metric.

It is also complete.

THEOREM 8.4 (THURSTON).

$$L(X, Y) = \sup \alpha \frac{\ell_Y(\alpha)}{\ell_X(\alpha)},$$

where α ranges over all simple close curve on S.

LEMMA. If α is a simple closed curve which is a short curve on X or dual to a short curve on X, then

$$L(X, Y) \stackrel{+}{\asymp} \max \frac{\ell_y(\alpha)}{\ell_x(\alpha)}.$$

($\stackrel{+}{\asymp}$ is $=$ up to additive error)

We do some examples of finding the Thurston distance between points in $T_{1,1}$.

On i, the length of α is i, and the length of α is $1/y$ on yi, thus

$$d_L(yi, i) \stackrel{+}{\asymp} \log(y).$$

On the other hand, by the collar lemma, the length of the blue curve is $\log(y)$, hence

$$d_L(i, y_i) \stackrel{+}{\asymp} \log(\log(y)).$$

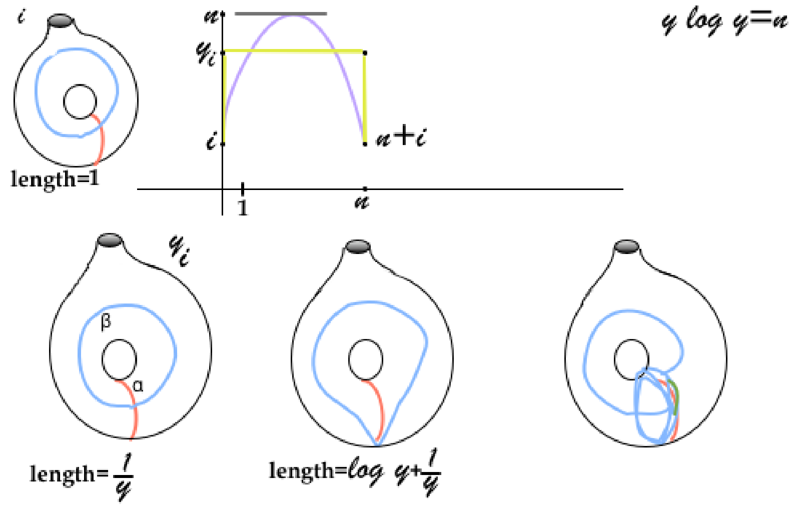

Figure 7: lengths on $T_{1,1}$

On $1 + yi$, the length of the blue curve is $\log(y) + \frac{1}{y}$, hence

$$d_L(yi, 1 + yi) \overset{+}{\asymp} \log(1 + \frac{1}{y \log y}) \overset{+}{\asymp} \frac{1}{y \log y}.$$

Now give a large integer n, let $y \log y = n$, so $d(y_i, n + y_i) \asymp 1$. We see that

$$d_L(i, y_i) + d(y_i, n + y_i) + d(n + y_i, n + i) \asymp \log n \asymp d_L(i, n + i).$$

9. Description of Geodesics

We can give the following description of geodesics $X, Y \in T_g$:

Definition 9.1. A map $h : X \to Y$ is called *extremal* if $K_h = L(X, Y)$.

Theorem 9.2 (Thurston). The set $\bigcap_{\text{h extremal}} \{\text{stretch locus of h}\}$ is a geodesic lamination $\lambda(X, Y)$, called the maximally-stretched lamination.

Remarks.

(i) $\text{Env}(X, Y) = \{\text{geodesics from } X \text{ to } Y\} \neq \emptyset$ but $|\text{Env}(X, Y)|$ can be infinite. Each element of $\text{Env}(X, Y)$ must stretch $\lambda(X, Y)$ maximally.

(ii) Elements in $\text{Env}(X, Y)$ do not necessarily fellow-travel, the reversal a geodesic from X to Y may not be a geodesic from Y to X, even after reparametrization (Lenzhen-Raf-T)

(iii) From the coarse perspective, the shadow map from T_g to the curve complex

$$T_g \to \mathcal{C}(S)$$

defined by sending X to a short curve on X sends every Thurston geodesic to a reparametrized quasi-geodesic in $\mathcal{C}(S)$ (LRT). The same statement is not true if we replace S by a proper subsurface of S.

Open Questions.

1. Are there preferred geodesics in $\mathrm{Env}(X, Y)$?

2. Is there a combinatorial description (in the sense of Rafi) of a Thurston geodesic? Is there a distance formula?

3. What does $\mathrm{Env}(X, Y)$ look like? In $T_{1,1}$, $\mathrm{Env}(X, Y)$ is the intersection of two cones; a complete understanding is in progress (Dumas-Lenzhen-Rafi-Tao).