LECTURE 2. Conjugate matrices

Let A be a hyperbolic matrix in $SL(n,\mathbb{Z})$ with irreducible polynomial f and hence distinct eigenvalues, $K=\mathbb{Q}(\lambda)$, where λ is an eigenvalue of A and $\mathcal{O}_K=\mathbb{Z}[\lambda]$.

Definitions

We say that $A, B \in SL(n,\mathbb{Z})$ are conjugate over $\mathbb{Z}$ (denoted $A \sim B$) if $\exists$ $C \in SL(n,\mathbb{Z})$ s.t. $B = C^{-1}AC$.
Two ideals I and J in $\mathcal{O}_K$ are equivalent if there exists non-zero $\alpha, \beta \in \mathcal{O}_K$ s.t. $\alpha I = \beta J$. The set of equivalence classes (ideal classes) forms a finite group, called the class group of $\mathcal{O}_K$ (or of K). Its order is called the class number, denoted by $h(K)$.

If $A \sim B$, they have the same characteristic polynomial and the same eigenvalues. To each matrix A' conjugate to A we assign an eigenvector $v=(v_1,\ldots,v_n)$ with eigenvalue λ: $A'v=\lambda v$ with all its entries in $\mathcal{O}_K$, and to this eigenvector, an ideal in $\mathcal{O}_K$ with the $\mathbb{Z}$-basis $v_1,\ldots,v_n$.

Conjugate matrices

It follows from an old Theorem of Latimer and MacDuffee [LM], see also [T] and a more modern account in [W].

Theorem

The described map is a bijection between conjugacy classes of hyperbolic elements in $SL(n,\mathbb{Z})$ with the same characteristic polynomial f and ideal classes in the order $\mathcal{O}_K = \mathbb{Z}[X]/(f(X))$.

[LM] C.G. Latimer and C.C. MacDuffee, *A correspondence between classes of ideals and classes of matrices*, Ann. Math. **74 (1933), 313–316.**

[T] O. Taussky, *Introduction into connections between algebraic number theory and integral matrices*, Appendix to: H. Cohn, A classical invitation to algebraic numbers and class field, Springer, New–York, 1978.

[W] D.I. Wallace, *Conjugacy classes of hyperbolic matrices in SL$(n,\mathbb{Z})$ and ideal classes in an order.* Trans. Amer. Math. Soc. **283** (1984), 177–184.

Case $n = 2$

Let $A \in SL(2,\mathbb{Z})$ be hyperbolic. Its characteristic polynomial $x^2 - \operatorname{tr}(A)x + 1$. If $B \in SL(2,\mathbb{Z})$ and $B \sim A$, then $\operatorname{tr}(B) = \operatorname{tr}(A)$. In this case K is a totally real quadratic field, and $Z(A)$ has rank one, i.e. if $B \in SL(2,\mathbb{Z})$ commutes with A, $B = A^k$ for some integer k.

Some hyperbolic geometry

The group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm 1_2\}$ acts on the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by fractional-linear (Möbius) transformations

$$z \mapsto \gamma(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}).$$

They are isometries of $\mathcal{H}$ with hyperbolic metric $ds = \frac{\sqrt{dx^2+dy^2}}{y}$.
Möbius transformations are classified by the number of fixed points in $\mathcal{H}$: $z = \frac{az+b}{cz+d}$, i.e., $cz^2 + (d-a)z - b = 0$, which depends on the value of the $|\operatorname{tr}(A)| = |a+d|$.
$M = PSL(2,\mathbb{Z})\backslash\mathcal{H}$ –modular surface (finite hyp. volume, non-comp.)

Closed geodesics on the modular surface

The bijection between conjugacy classes of hyperbolic elements and ideal classes in the corresponding real quadratic field extends further:

- If $A \sim B$, $\mathrm{tr}\,(A) = \mathrm{tr}\,(B) = t$. The axes of Möbius transformations corresponding to A and B become the same closed geodesic on M of length $2\log\frac{t+\sqrt{t^2-4}}{2}$.
- Conjugacy classes of hyperbolic elements in $PSL(2,\mathbb{Z})$ of a given trace $\Longleftrightarrow$ ideal classes in $\mathcal{O}_K \Longleftrightarrow$ closed geodesics on M of the same length $\Longleftrightarrow$ congruence classes of primitive integral indefinite quadratic forms of the corresponding discriminant.

Questions:

(1) How many geodesics of a given length are on the modular surface? $=$ How many non-conjugate over $\mathbb{Z}$ matrices with the same trace are in $SL(2,\mathbb{Z})$?

(2) How to find out if two matrices with the same trace are conjugate over $\mathbb{Z}$?

The answers can be found in [K] and [KU]: Relation to quadratic forms and quadratic fields

- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow Q_A(x, y) = cx^2 + (d-a)xy - by^2$
- $SL(2, \mathbb{Z})$ acts on quadratic forms by substitutions: for $C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$, let $x = \alpha x' + \beta y'$, $y = \gamma x' + \delta y'$ and define $Q' = C \cdot Q$ by $Q'(x, y) = Q(x', y')$.
- We say $Q' \sim Q$ if $Q' = C \cdot Q$ for some $C \in SL(2, \mathbb{Z})$.
- $|\mathrm{tr}\, A| > 2 \Longrightarrow D = (a+d)^2 - 4 > 0$ (it is easy to see that D is not a perfect square), so Q_A is an integral indefinite quadratic form.
- $A \sim B \Longleftrightarrow Q_A \sim Q_B$ (in narrow sense, i.e. via a matrix from $SL(2, \mathbb{Z})$).
- Class number $h(D)$ (in narrow sense): number of non-equivalent quadratic forms with given discriminant.

[K] S. Katok, *Coding of closed geodesics after Gauss and Morse*, Geom. Dedicata, **63** (1996), 123–145.
[KU] S. Katok and I. Ugarcovici, *Symbolic dynamics for the modular surface and beyond*, Bull. of the Amer. Math. Soc., **44**, no. 1 (2007), 87-132.

Relation to quadratic forms and quadratic fields

Does the relation go the other way?

- Let $Q(x,y) = Pz^2 + Qz + R$ be an integral quadratic form with $D = Q^2 - 4PR > 0$ not a perfect square. Consider a geodesic γ connecting the real roots of the quadratic equation $Q(z,1) = 0$.
- The set of all rational matrices having γ as their axis is a real quadratic field $K = \mathbb{Q}(\sqrt{D}) = \{\lambda\alpha + \mu,\ \lambda \in \mathbb{Q}^*, \mu \in \mathbb{Q}\}$, where $\alpha \in M(2,\mathbb{Z})$ is some matrix with the axis γ, e.g. $\alpha = \begin{pmatrix} 0 & -R \\ P & Q \end{pmatrix}$ (hence the discriminant of the characteristic equation for α is equal to D).
- Determinant matrices are equals to the norms of corresponding elements in K.
- Matrices in K that belong to $M(2,\mathbb{Z})$ correspond to the ring of integers in K.
- Is there a matrix in K that belongs to $SL(2,\mathbb{Z})$?
  Yes, it corresponds to a non-trivial unit in K of norm 1.

Gauss reduction theory in matrix language

Definition

A hyperbolic matrix $A \in SL(2, \mathbb{Z})$ is caled reduced if its attracting and repelling fixed points w and u satisfy $w > 1, \ \ 0 < u < 1$.

Theorem (Reduction Algorithm)

There is a finite number of reduced matrices in $SL(2, \mathbb{Z})$ with given trace $t, \ |t| > 2$. Any hyperbolic matrix in $SL(2, \mathbb{Z})$ with trace t can be reduced by a finite number of standard conjugations. Applied to a reduced matrix, it gives another reduced matrix. Any reduced matrix conjugate to A is obtained from A by a finite number of standard conjugations. Thereby the set of reduced matrices is decomposed into disjoint cycles of conjugate matrices.

The notion of reduced and standard conjugations are related to a particular theory of continued fractions: minus continued fractions.

Minus continued fractions

Any real number x can be written uniquely in the form of a minus continued fraction:

$$x = n_0 - \cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots}}} = \lceil n_0, n_1, \cdots, \rceil$$

where $n_0 = \lceil x \rceil = \lfloor x \rfloor + 1$, $x_1 = -\frac{1}{x - n_0}$; $n_{i+1} = \lceil x_{i+1} \rceil$, $x_{i+1} = -\frac{1}{x_i - n_i}$, i.e. the sequence $r_k = \lceil n_0, n_1, \ldots, n_k \rceil$ converges to x.
Conversely, any sequence of integers $n_0, n_1, \ldots,$ where $n_i \in \mathbb{Z}$ and $n_i \geq 2$ for $i \geq 1$ defines a minus continued fraction as above.

The theory is similar to that of ordinary continued fractions which has $+$'s instead of $-$'s and $\lfloor \cdot \rfloor$ instead of $\lceil \cdot \rceil$, but is more convenient for our purposes.

Properties of minus continued fractions

are very similar to these of ordinary continued fractions:

(1) For the ordinary continued fractions, rational numbers have finite expansions. For minus continued fractions expansions are always infinite: for rational numbers they have tails of 2's.

(2) A number is a quadratic irrationality iff its expansion is eventually periodic. This also holds for ordinary continued fractions.

(3) α has a purely periodic minus continued fraction expansion iff α is a quadratic irrationality, $\alpha > 1$ and $0 < \alpha' < 1$, where α' is number conjugate to α. These are inequalities that appeared in the definition of reduced matrix. For ordinary continued fractions a definition of reduced (in wide sense) is used.

(4) $\alpha = C\beta$ (connected by a Möbius transformation in $C \in SL(2,\mathbb{Z})$) iff the periods of expansions of α and β differ by a cyclic permutation. For ordinary continued fractions this holds with $GL(2,\mathbb{Z})$ in place of $SL(2,\mathbb{Z})$.

Solution to the conjugacy problem

Theorem [K]

Two hyperbolic matrices A and B in $SL(2,\mathbb{Z})$ with the same trace are conjugate over $\mathbb{Z}$ iff the periods in the minus continued fraction expansions of their attracting fixed points w_A and w_B are cyclic permutations of one another.

Proof: (a) If A and $B \in SL(2,\mathbb{Z})$ have a common fixed point, then their second fixed points also coincide. This follows from discreteness of the group $PSL(2,\mathbb{Z})$ in $PSL(2,\mathbb{R})$.
(b) If periods of w_A and w_B differ by a cyclic permutation, there exists a $C \in SL(2,\mathbb{Z})$ such that $w_A = Cw_B$ by (4). Then the matrices CBC^{-1} and A have the same fixed point w_A, and by (a), since they have the same trace, either $CBC^{-1} = A$ or $CBC^{-1} = A^{-1}$. Since both w_A and w_B are attracting, w_A is attracting for A and CBC^{-1}, hence $CBC^{-1} = A$. (c) If $A \sim B$, $CBC^{-1} = A \Rightarrow w_A = Cw_B$, and by (4) periods of w_A and w_B differ by a cyclic permutations.

[K] S. Katok, *Coding of closed geodesics after Gauss and Morse*, Geom. Dedicata, **63** (1996), 123–145.

Examples

- Since $h(5) = 1$, all matrices in $SL(2,\mathbb{Z})$ with trace 3 are conjugate over $\mathbb{Z}$.
- Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}$.
  Which are conjugate over $\mathbb{Z}$?
  $w_A = \lceil 1, 4, 4, \dots \rceil$, period (4); $w_B = \lceil 1, 3, 2, 3, 2 \dots \rceil$, period $(3,2)$; $w_D = \lceil 0, 4, 4, \dots \rceil$, period (4). Thus $A \not\sim B, A \sim D, B \not\sim D$.
  Incidentally, $h(12) = 2$.

If $A, B \in SL(2,\mathbb{Z})$ have the same characteristic polynomial, hence the same eigenvalues, they are conjugate over $\mathbb{Q}$. If they are non-conjugate over $\mathbb{Z}$: $A \not\sim B$, the automorphisms of $\mathbb{T}^2$, T_A and T_B are not algebraically isomorphic , but their entropies are equal: $h_\mu(T_A) = h_\mu(T_B) = \log|\lambda|$, where λ is the eigenvalue with $|\lambda| > 1$), and, being Bernoulli, these automorphisms are measurably conjugate with respect to the Lebesgue measure μ.

Higher rank $n > 2$: measure rigidity implications

Higher rank $n > 2$: measure rigidity implications

The situation for $n > 2$ is dramatically different due to a so-called measure rigidity. The counterparts of hyperbolic automorphism of $\mathbb{T}^2$ and all its integral powers ($\mathbb{Z}$-action) are Cartan actions of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$. They are generated by maximal rank abelian semisimple subgroups of $SL(n, \mathbb{Z})$. Measure rigidity for Cartan actions implies, in particular, that such actions are measurably conjugate only if they are algebraically conjugate over $\mathbb{Z}$.
The following theorem generalizes Latimer-McDuffee theorem to centralizers.

Higher rank $n > 2$: matrices with non-conjugate centralizers

Theorem [KKS]

Let $A \in SL(n, \mathbb{Z})$ be a hyperbolic matrix with irreducible characteristic polynomial f and distinct real eigenvalues, $K = \mathbb{Q}(\lambda)$ where λ is an eigenvalue of A, and $\mathcal{O}_K = \mathbb{Z}[\lambda]$. Suppose the number of eigenvalues among $\lambda_1, \dots, \lambda_n$ that belong to K is equal to r. If the class number $h(K) > r$, then there exists a matrix $A' \in SL(n, \mathbb{Z})$ having the same eigenvalues as A whose centralizer $Z(A')$ is not conjugate in $GL(n, \mathbb{Z})$ to $Z(A)$. Furthermore, the number of matrices in $SL(n, \mathbb{Z})$ having the same eigenvalues as A with pairwise nonconjugate (in $GL(n, \mathbb{Z})$) centralizers is at least $[\frac{h(K)}{r}] + 1$, where $[x]$ is the largest integer $< x$.

Matrices with non-conjugate centralizers produce actions that are not algebraically isomorphic even up to a time change, and hence are not measurably isomorphic.

[KKS] A. Katok, S. Katok and K. Schmidt, *Rigidity of measurable structure for $\mathbb{Z}^d$ actions by automorphisms of a torus*, Comment. Math. Helv., **77**, no. 4 (2002), 718–745

Example of non-isomorphic Cartan actions

Let K be a totally real cubic field with class number 3, the Galois group S_3 and discriminant 2597. It can be represented as $K = \mathbb{Q}(\lambda)$ where λ is a unit in K with minimal polynomial $f(x) = x^3 - 2x^2 - 8x + 1$. In this field the ring of integers $\mathcal{O}_K = \mathbb{Z}[\lambda]$, and the fundamental units are $\lambda_1 = \lambda$ and $\lambda_2 = \lambda + 2$.

Multiplications by λ_1 and λ_2 generate actions on three different lattices, $\mathcal{O}_K$ with the basis $\{1, \lambda, \lambda^2\}$, representing the principal ideal class, $\mathcal{L}$ with the basis $\{2, 1+\lambda, 1+\lambda^2\}$ representing the second ideal class, and $\mathcal{L}^2$ with the basis $\{4, 3+\lambda, 3+\lambda^2\}$ representing the third ideal class: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 8 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 8 & 4 \end{pmatrix}$; $\begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ -6 & 9 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ -6 & 9 & 4 \end{pmatrix}$; $\begin{pmatrix} -3 & 4 & 0 \\ -3 & 3 & 1 \\ -10 & 11 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 4 & 0 \\ -3 & 5 & 1 \\ -10 & 11 & 4 \end{pmatrix}$.

They are not algebraically isomorphic even up to a time change, and therefore not measurably isomorphic.

Discussion of further questions for higher rank $n > 2$

Let A be a hyperbolic matrix in $SL(n, \mathbb{Z})$ with irreducible polynomial f and hence distinct eigenvalues, $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of A and $\mathcal{O}_K = \mathbb{Z}[\lambda]$. Then the axes of $Z(A) = \mathbb{Z}^{n-1}$ in the factor $SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R})$ define a torus (or a "flat"). The number of different flats corresponding to matrices conjugate to A over $\mathbb{Z}$ is equal to the class number $h(K)$. The volume of each flat equal to kR_K, where $k = [U_K : \gamma(Z(A)]$.

Main question:

How to find out if two matrices in $SL(n, \mathbb{Z})$ with the same characteristic polynomial are conjugate over $\mathbb{Z}$? The answer should lead to a theory of multidimensional continued fractionsand the related reduction theory.