# PATTERNS IN PRIMES AND DYNAMICS ON NILMANIFOLDS

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## 1. Characteristic factors

Recall the setting from the first lecture of examining one equation in 3-variables. The test case we used was 3-term arithmetic progressions (APs) and we did this in primes and sets of positive density using the circle method and ergodic theoretic arguments.

The combinatorial argument stated that a lack of many 3-term APs, i.e $< \frac{\delta^3 N^2}{2}$, indicated that

$$\left\{ \sum_{x \le N} (\mathbb{1}_E(x) - \delta e(\alpha x)) \right\} \gg_\delta N$$

where $E \subset [N]$ and $|E| = \delta N$. Define the following averaging function $\mathbb{E}_{x \le N} := \frac{1}{N}$ then we obtain the equivalent formula

$$|\mathbb{E}_{n \le N} (\mathbb{1}_E(x) e(\alpha x))| \gg_\delta 1.$$

The ergodic theoretic argument gave that either

$$\frac{1}{N} \sum \mu(A \cap T^{-n} A \cap T^{-2n} A) \to (\mu(A))^3$$

where $\mu(A) = \delta$, or

$$\left| \int (\mathbb{1}_A - \delta) \psi(x) \mathrm{d}\mu \right| > 0$$

for some eigenfunction $\psi$. We can calculate the asymptotics,

$$\begin{aligned} \frac{1}{N} \sum \mu(A \cap T^{-n} A \cap T^{-2n} A) &\sim \frac{1}{N} \sum \int \pi_* \mathbb{1}_A(z) \pi_* \mathbb{1}_A(z + n\alpha) \pi_* \mathbb{1}_A(z + 2n\alpha) \mathrm{d}z \\ &= \int \pi_* \mathbb{1}_A(z) \pi_* \mathbb{1}_A(z + b) \pi_* \mathbb{1}_A(z + 2b) \mathrm{d}z \mathrm{d}b \end{aligned}$$

where $\pi : X \to Z$ is a projection of the function to the group rotation factor $Z$. In Furstenburg's proof of Szemeredi's theorem, he found the asymptotics for the average of finding $(k+1)$-term arithmetic progressions:

$$\frac{1}{N} \sum \mu(A \cap T^{-n} A \cap \cdots \cap T^{kn} A) \sim \frac{1}{N} \sum \int \pi_* \mathbb{1}_A(y) \cdots \pi_* \mathbb{1}_A(T^{kn} y) \mathrm{d}\pi_* \mu \tag{1}$$

with $\pi : X \to Y$ a projection to a tower of isometric extensions $Y = * \times Z_1 \times_{\sigma_1} M_1 \times_{\sigma_2} M_2 \times \cdots \times_{\sigma_{k-2}} M_{k-1}$. At step $j$ we have $Z_j = Z_{j-1} \times_{\sigma_j} M_j$, $\sigma_j : Z_{j-1} \to \mathrm{Isom}(M_j)$. Furstenburg also showed that you can replace the characteristic function $\mathbb{1}_A$ with any $k$-tuple of functions.

**Definition 1.** If $\pi : X \to Y$ and (1) is satisfied, then we call $Y$ a *characteristic factor* for $(k+1)$-term progressions.

We'd like to find a good characteristic factor. For example if $k = 1$, the ergodic theorem gives

$$\frac{1}{N} \int f_0(x) f_1(T^n x) \to \int f_0 \int f_1$$

and $Y = \{*\}$ is a characteristic factor. For $k = 2$, Furstenburg's argument gives that the Kronecker factor is characteristic.

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**Example 1** (Why abelian factors are not characteristic for 4-term progressions)**.** Let $X = \mathbb{T}^2$, and look at the map $(x, y) \mapsto (x + \alpha, y + x)$. Consider the function $\varphi(x, y) = e(y)$, a 2-step eigenfunction. This construction allows us to find a counterexample. Take $f_0 = \varphi$, $f_1 = \bar{\varphi}^3$, $f_2 = \varphi^3$, $f_4 = \varphi^{-1}$, then since

$$T^n\varphi(x, y) = e\left(\binom{n}{\alpha}\alpha\right)e(nx)e(y)$$

we have $\varphi T^n\varphi^3T^{2n}\varphi^{-3}T^{3n}\varphi^{-1} \equiv 1$ and so

$$\frac{1}{N}\sum\int\varphi T^n\varphi^3T^{2n}\varphi^{-3}T^{3n}\varphi^{-1} = 1$$

but this is a contradiction since $\langle\varphi, \psi\rangle = 0$ for any eignefunciton $\psi$ so the projection of $\varphi$ on the Kronecker factor is 0 and we are unable to use Furstenberg's argument to find the asymptotics.

**Definition 2.** A factor $Y$ of $X$ is *universal* for $k$-term progressions if for any $W$, $k$-characteristic, the factor map $X \to Y$ factors through $W$.

**Definition 3.** Let $N$ be a 2-step nilpotent Lie group and $\Gamma$ a lattice. Fix $a \in N$, then $(N/\Gamma, \mathcal{B}, \text{Haar}, a)$ is a 2-step *nilsystem.*

**Definition 4.** A *pro-nilsystem* is the inverse limit $\varprojlim(N_i/\Gamma_i, a_i)$.

**Theorem 1** (Furstenberg-Weiss, Conze-Lesigne)**.** *The universal characteristic factor for 4-term arithmetic progressions is a 2-step pro-nilsystem.*

**Theorem 2** (Host-Kra 2005, Ziegler 2007)**.** *The universal characteristic factor for $k$-APs is a $(k-2)$-step pro-nilsystem*

**Corollary 1.** *We get the following asymptotic:*

$$\frac{1}{N}\sum\int f_0(x)\cdots f_k(T^{kn}x)d\mu \longrightarrow \int f_0(x)\cdots f_k(x_k)d\nu$$

*where $\nu$ is Haar measure on a nice subnilmanifold of $(N/\Gamma)^{k+1}$.*

## 2. Gower's argument: generalizing Roth's proof

**Observation 1.** Averages for arithmetic progressions are "controlled" by more symmetric forms

**Definition 5** (Gower's $U_k$ norm)**.** For $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, set $\Delta_h f(x) = f(x+h)\overline{f(x)}$, $\|f\|_{U1} = |\mathbb{E}_{x\leq N}f(x)$ and inductively define $\|f\|_{U_k}^{2^k} = \mathbb{E}_{h\leq N}\|\Delta_h f(x)\|_{U_{k-1}}^{2^{k-1}}$.

**Example 2.** For $k = 2$,

$$\|f\|_{U_2} = \mathbb{E}_{h\leq N}\|\Delta_h f(x)\|_{U_1}^2 = \mathbb{E}_{x,h,k\leq N}f(x)\overline{f(x+h)f(x+k)}f(x+h+k) = \mathbb{E}_{x,h,k\leq N}\Delta_k\Delta_h f(x).$$

We have control over the size of these averages:

$$|\mathbb{E}_{x,d\leq N}f_0(x)f_1(x+d)f_2(x+2d)| \leq \|f_i\|_{U_2}.$$

for $i = 0, 1, 2$. In particular,

$$\left|\mathbb{E}_{x,d\leq N}\mathbb{1}_E(x)\mathbb{1}_E(x+d)\mathbb{1}_E(x+2d) - \delta^3\right| \leq 3\|\mathbb{1}_E - \delta\|_{U_2}.$$

We can generalize this inequality to work for any $k$:

$$\left|\mathbb{E}_{x,d\leq N}\mathbb{1}_E(x)\cdots\mathbb{1}_E(x+dk) - \delta^{k+1}\right| \leq k\|\mathbb{1}_E - \delta\|_{U_k}$$

If we have too few $(k+1)$-term APs, then $|\mathbb{1}_E(x)\mathbb{1}_E(x+d)\mathbb{1}_E(x+2d) - \delta^3| > \frac{\delta^3}{2}$ and we have $\|\mathbb{1}_E - \delta\|_{U_2} > \frac{\delta^3}{6}$. So in general, we either have lots of $k$-term APs, or $\|\mathbb{1}_E - \delta\|_{U_k} \gg 1$.

**Question:** What can you say about functions $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{D}$ for which $\|f\|_{U_k} \gg_\delta 1$? For $k = 2$ you can use the circle method or discrete Fourier analysis to show that $f$ correlates with $|\sum f(x)e(\alpha x)| \gg_\delta N$.

**Theorem 3** (Gowers)**.** *If $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{D}$ with $\|f\|_{U_k} \gg_\delta 1$ then there is a partition of $[N]$ into APs $P_i$, $|P_i| \geq N^{\alpha(\delta,k)}$, and polynomials $p_i(x)$, of degree $< k$, such that*

$$\sum_i \left| \sum_{x \in P_i} f(x) e(p_i(x)) \right| \gg_\delta N$$

*(for example, for 4-term APs, the $p_i$ are quadratic).*

We can use this argument to find a long AP of size $\gg N^{\tilde{\alpha}(\delta,k)}$ on which $E$ has increased density.

## 3. The Inverse Theorem for Gowers Norms

It turns out that the obstructions to Gowers uniformity norms come from nilmanifolds:

**Theorem 4** (Green-Tao-Ziegler)**.** *Suppose $f : [N] \to \mathbb{D}$ and $\|f\|_{U_k} \geq \delta$, then there exists $G/\Gamma$ a $(k-1)$-step nilmanifold of dimension $\ll_\delta 1$, a function $F : G/\Gamma \to \mathbb{D}$ with $\|F\|_{Lip} \ll_\delta 1$ and $a \in G$ such that*

$$\left| \sum_{x \leq N} f(x) F(a^x \Gamma) \right| \gg_\delta N. \tag{2}$$

*Conversely, if (2) holds, then $\|f\|_{U_k} \gg_\delta 1$.*

We call $F(a^x\Gamma)$ a $(k-1)$-step nilsequence.

**Remark 1.** This works for any system of affine linear forms $L_1(\vec{n}), \ldots, L_k(\vec{n})$, $L_i(\vec{n}) = \sum_{j=1}^{M} a_i n_j + b_i$ so long as no two of the $L_i$ are affinity dependent. That is, we have

$$|\mathbb{E} f_1(L_1(\vec{n}) \cdots f_k(L_k(\vec{n}))| \leq \|f_i\|_{U_{J(L_1,\ldots,L_k)}}.$$

### 3.1. The Möbius function.

Let $n = p_1 \cdots p_k$ where $p_i$ are distinct primes, then

$$\mu(n) = \begin{cases} 1 & \text{for } k \text{ odd} \\ -1 & \text{for } k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 5** (Green-Tao)**.** *Let $g(n)$ be a nilsequence of bounded complexity and*

$$\frac{1}{N} \sum \mu(n) g(n) \ll_J \frac{1}{(\log N)^J}$$

*then we have,*

$$\frac{1}{N} \sum \mu(L_1(\vec{n})) \cdots \mu(L_k(\vec{n})) = o(1).$$

**Remark 2.** This result can be pushed to calculate $\mathbb{1}_{\mathbb{P}}$.