Introduction to Ratner's Theorems on Unipotent Flows

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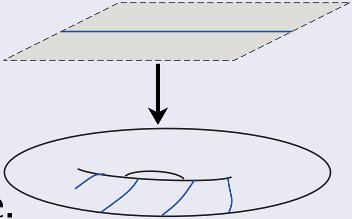
Abstract. Let f be the obvious covering map from Euclidean n-space to the n-torus. It is well known that if L is any straight line in n-space, then the closure of $f(L)$ is a very nice submanifold of the n-torus. In 1990, Marina Ratner proved a beautiful generalization of this observation that replaces Euclidean space with any Lie group G, and allows L to be any subgroup of G that is "unipotent." We will discuss the statement of this theorem and related results, some of the ideas in the proofs, and a few of the important consequences.

Elementary example

Let M = torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

- covering map $f\colon \mathbb{R}^2 \to M$
- H = line in $\mathbb{R}^2$.

If the slope of H is irrational,
it is classical that $f(H)$ is dense.



Exercise. Let M = n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

- covering map $f\colon \mathbb{R}^n \to M$
- H = vector subspace of $\mathbb{R}^n$.

Closure $\overline{f(H)} = f(S)$ is a torus $\mathbb{T}^k$ ($\exists$ subspace S of $\mathbb{R}^n$).

The closure of $f(H)$ is a very nice submanifold of M.

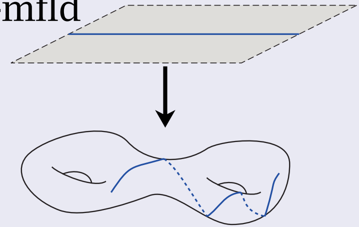
Example in Riemannian geometry

Example

Let M = compact, hyperbolic n-mfld

- covering map $f\colon \mathbb{H}^n \to M$
- line $\mathbb{H}^1 \hookrightarrow \mathbb{H}^n$

Closure $\overline{f(\mathbb{H}^1)}$ can be a fractal.



Consequence of Ratner's Thm [Shah, Payne]

$\mathbb{H}^2 \subset \mathbb{H}^n \implies \overline{f(\mathbb{H}^2)} = f(\mathbb{H}^k)$ is a submfld of M

(immersed, maybe not embedded).

(Similar for other locally symmetric spaces.)

Recall

- $f\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$
- H = vector subspace of $\mathbb{R}^n$

$\implies \overline{f(H)} = f(S)$, $\exists$ vector subspace S of $\mathbb{R}^n$.

- $\mathbb{R}^n$ is a **Lie group** (group & manifold)
- subgroup $\mathbb{Z}^n$ is a **lattice**
  (discrete and $\mathbb{R}^n/\mathbb{Z}^n$ has finite volume)

Generalization (Ratner's Theorem) [1991]

Replace:

- $\mathbb{R}^n$ with any Lie group G
- $\mathbb{Z}^n$ with any lattice Γ in G
- H with any subgroup of G
  that is generated by "unipotent" elements
- S with a closed subgroup of G

Homogeneous Dynamics

study of dynamical systems on homogeneous spaces

finite-volume homogeneous space G/Γ:

- G = Lie group = closed subgroup of $\mathrm{SL}(n,\mathbb{R})$
  $\{\, n \times n$ mats with $\mathbb{R}$ entries, $\det = 1 \,\}$
  = group & manifold
- Γ = ~~closed subgroup of G~~ lattice in G

Coset space G/Γ is a manifold of finite volume.

"dynamical system" = action of subgroup H of G

$h \colon G/\Gamma \to G/\Gamma \qquad h(x\Gamma) = hx\Gamma$

E.g., understand the orbit $Hx\Gamma$ in G/Γ

Ratner's Theorem [1991]

- finite-volume homogeneous space G/Γ
- subgroup H gen'd by **unipotent** elements

$\Rightarrow \overline{Hx\Gamma} = Sx\Gamma$ for some closed subgroup S of G.

Also: $H \subseteq S$ and $(x\Gamma x^{-1}) \cap S$ is latt in S if H conn.

Unipotent matrices are conjugate to an element of

$$\left\{ \begin{bmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} \right\} \subset \mathrm{SL}(n, \mathbb{R}).$$

Exer. u unip $\Leftrightarrow (u - I)^n = 0 \Leftrightarrow$ char poly $(x-1)^n$
$\Leftrightarrow$ only eigenvalue is 1 $\Rightarrow$ *not* diag'ble (unless $u = I$).

Applications of Ratner's Theorem

Example (Shah [1991], Payne [1999])

$M = \mathbb{H}^n/\Gamma, \quad f\colon \mathbb{H}^n \to M, \quad \mathbb{H}^2 \subset \mathbb{H}^n$

$\implies \overline{f(\mathbb{H}^2)}$ is (immersed) submanifold of M.

Idea of proof.

- $\mathbb{H}^n = K \backslash \mathrm{SO}(1,n)^\circ = K \backslash G \quad \Rightarrow \pi\colon G/\Gamma \to M$
- $f(\mathbb{H}^2) = \pi\big(\mathrm{SO}(1,2)^\circ x\Gamma\big) = \pi(Hx\Gamma)$

$\overline{f(\mathbb{H}^2)} = \overline{\pi(Hx\Gamma)} = \pi\big(\overline{Hx\Gamma}\big) \overset{\text{Ratner}}{=} \pi(Sx\Gamma)$

= immersed submanifold. □

$H = \mathrm{SO}(1,2)^\circ \cong \mathrm{SL}(2,\mathbb{R}) = \begin{bmatrix} * & * \\ * & * \end{bmatrix} = \left\langle \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \right\rangle$

is generated by unipotent elements

"Oppenheim Conjecture" (Margulis [1987])

Let Q be a real quadratic form in $n \geq 3$ variables
(e.g., $x^2 - \sqrt{2}xy + \sqrt{3}z^2$).
Then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$
unless $\approx$ $\mathbb{Z}$-coefficients, or definite, or degenerate.

Proof for $n = 3$.

Let $G = \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, and
$H = \mathrm{SO}(Q) = \{\, h \in \mathrm{SL}(3,\mathbb{R}) \mid Q(h\vec{x}) = Q(\vec{x}) \,\}$.
Ratner: $\overline{H\Gamma} = S\Gamma$, for some subgroup $S \supseteq H$.
Algebra: H is maximal in G, so $S = H$ or G.
$S = H \implies Q$ has $\mathbb{Z}$-coefficients ($\approx$)
So $H\Gamma$ is dense in G. Therefore
$\overline{Q(\mathbb{Z}^3)} \supset Q\big(\overline{H\Gamma}\mathbb{Z}^3\big) = Q(G\mathbb{Z}^3) = Q(\mathbb{R}^3) = \mathbb{R}$. □

Example (Shah [1998])

Γ, Λ lattices in $G = \mathrm{SL}(n, \mathbb{R})$ (any simple Lie group)
$\Rightarrow$ every Λ-orbit on G/Γ is either finite or dense.

Proof. Ratner: $\overline{\Lambda x \Gamma} = S x \Gamma$, and $S \supseteq \Lambda$.
Borel Density Theorem: $\Lambda \not\subset$ conn, proper subgrp.
$\therefore$ Λ normalizes H connected $\Rightarrow$ $N_G(H) = G$
$\Rightarrow$ (bcs G simple) $H = \{e\}$ or G.
$S^\circ = \{e\} \Rightarrow S/\Lambda$ finite. $S^\circ = G \Rightarrow S/\Lambda = G/\Lambda$. □

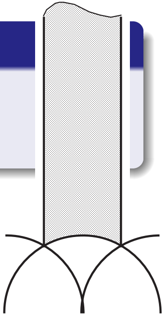
Gap: Ratner's Thm requires Λ to be gen'd by unips.

Exer. Fix by using fact that G is gen'd by unips.
Hint: Look at orbit of $\{(g,g)\}$ on $(G \times G)/(\Gamma \times \Lambda)$.
G simple $\Rightarrow \{(g,g)\}$ is max'l conn subgrp.

A Key Idea in the Proof

Example

$G = \mathrm{SL}(2,\mathbb{R}) = \{\ 2 \times 2$ real mat's of det $1\ \}$.
Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Then Γ is a lattice in G.



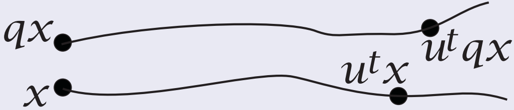
Other choices of Γ can make G/Γ compact.

Definition

Define $\quad u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad$ and $\quad a^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$.

Each is a homomorphism from $\mathbb{R}$ to $\mathrm{SL}(2,\mathbb{R})$.
u^t is a **unipotent** one-parameter subgroup.

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$



$d(x, qx) = \|q\|$. $\qquad d(u^t x, u^t q x) = \|u^t q u^{-t}\|$.

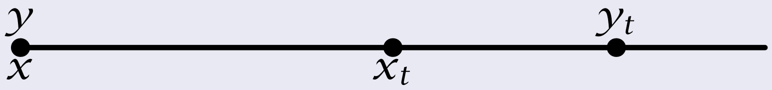
$$u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha) t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Shearing: Fastest motion is parallel to the orbits.

Cor. If $x \approx y$, then $\exists t$, $y_t \approx x_{t+1}$.

Shearing

Fastest motion is parallel to the orbits.



Contrast: $a^t q a^{-t} = \begin{bmatrix} \alpha & \beta e^{2t} \\ \gamma e^{-2t} & \delta \end{bmatrix}$

Fastest motion is transverse to the orbits.

Further reading (and references to primary sources)

chapter of **forthcoming book on arithmetic grps**
free PDF file on my web page (or the arxiv)
http://people.uleth.ca/~dave.morris
/books/IntroArithGroups.html

my book: ***Ratner's Theorems on Unipotent Flows***
free PDF file on my web page (or the arxiv)
http://people.uleth.ca/~dave.morris
/books/Ratner.html
published by University of Chicago Press (2005)
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