HOMOGENEOUS FLOWS AND THE STATISTICS OF DIRECTIONS

JENS MARKLOF

1. Review

So far we have seen that we can translate the question of the statistical distribution of the distances in the point set $\mathcal{P}$ into the more general question of the distribution of the point process $\Xi_T = \mathcal{P}K(v)D(T)$, with $K(v) \in \mathrm{SO}(d)$, $v \in S_1^{d-1}$ random according to λ, and $D(T) = \mathrm{diag}(T^{-1}, T^{d-1}, \mathbb{1}_{d-1})$. If we can prove that the sequence of point processes converge in finite dimensional distribution to a limiting process then we also understand the statistics of the directions, namely the cone.

Our main example was to take $\mathcal{P} = \mathcal{L} = \mathbb{Z}^d M$, a Euclidean lattice. There we saw that we could use the equidistribution of $\{\Gamma M K(v) D(T), v \in S_1^{d-1}\}$ in $\Gamma\backslash G$ and the Siegel Veech formula to prove that, at $T \to \infty$, $\Xi_T \Longrightarrow \Xi = \mathbb{Z}^d x, x \in \Gamma\backslash G$, x random according to μ.

2. Quasicrystals

2.1. **Setup.** The quasicrystals we will study are realized by a "cut-and-project" construction, one example of such is the class of Penrose tilings where you take your point set to be the vertices of of the tiling. Some quasicrystals cannot be obtained by this construction and it would be interesting to see if this theory can be extended to these.

Pick $m \geq 0$, $n = d + m$. Define two orthogonal projections:

$\pi : \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ the *physical space*
$\pi_{\mathrm{int}} : \mathbb{R}^n \to \mathbb{R}^m$ the *internal space*

Let $\mathcal{L} \in \mathbb{R}^n$ be a lattice of full-rank in $\mathbb{R}^n$, $\mathcal{A} = \overline{\pi_{\mathrm{int}}(\mathcal{L})} \leq \mathbb{R}^m$ abelian. Denote by $\mathcal{A}^0$ the connect component containing 0, $\dim \mathcal{A}^0 =: m_1 \leq m$. Then we can write $\mathcal{A} = \mathcal{A}^0 \oplus \pi(a_1)\mathbb{Z} \oplus \cdots \oplus \pi(a_{m_2})\mathbb{Z}$, where $m = m_1 + m_2$. There is a Haar measure on $\mathcal{A}$, $\mu_{\mathcal{A}}$.

2.2. **"Cut-and-project" construction.** Choose $W \subset \mathcal{A}$ a regular *window set.* Assume it is bounded and $\mu_{\mathcal{A}}(\partial W) = 0$. Take all lattice points whose internal projection falls in W, then project down to $\mathbb{R}^d$.

$$\mathcal{P}(W, \mathcal{L}) = \mathcal{P} = \{\pi(\ell) \mid \ell \in \mathcal{L}, \pi_{\mathrm{int}}(\ell) \in W\}$$

With a particular choice of $\mathcal{L}$ and W you get the vertex set of a Penrose tiling. Assume W, $\mathcal{L}$ chosen such that

$$\{\ell \in \mathcal{L} \mid \pi_{\mathrm{int}}(\ell) \in W\} \to \mathcal{P}$$

is bijective.

Theorem 1 (Hof '98, Schlottmann '98)**.**

$$\lim_{T\to\infty} \frac{\#(\mathcal{P} \cap B_T^d)}{vol(B_T^d)} = \frac{\mu_{\mathcal{A}}(W)}{vol(V/\mathcal{L} \cap V)}$$

where $V = \mathbb{R}^d \times \mathcal{A}^0$.

Date: February 03, 2015.

We want to understand the closure of the $\mathrm{SL}(d, \mathbb{R})$ orbit of $\mathcal{P}$. To do this, we follow the same process as in the case when $\mathcal{P}$ is the Euclidean lattice.

Define $G' = \mathrm{SL}(n, \mathbb{R})$, $\Gamma' = \mathrm{SL}(n, \mathbb{Z})$, $n = d + m$. For any $g \in G'$, define the embedding $\varphi_g : G \to G'$ where $A \mapsto g \begin{pmatrix} A & \\ & I_m \end{pmatrix} g^{-1}$.

Ratner's theorem tells us that for all $g \in G'$, there exists a (unique) closed, connected subgroup $H_g \leq G'$ such that:

(1) $\Gamma' \cap H_g$ is a lattice in H_g
(2) $\varphi_g(G) \subset H_g$
(3) $\overline{\Gamma' \backslash \Gamma' \varphi_g(G)} \simeq (\Gamma' \cap H_g) \backslash H_g$

Let $\mu_g := \mu_{H_g}$ be the unique H_g-invariant probability measure on $\Gamma' \backslash \Gamma' H_g$.

Theorem 2 (Marklof-Strömbergsson, follows from work of Shah and Ratner). *Assume $\lambda \ll \omega = vol_{S_1^{d-1}}$. Fix $g \in G'$. Then for every bounded, continuous function $f : \Gamma' \backslash \Gamma' H_g \to \mathbb{R}$,*

$$\lim_{T \to \infty} \int_{S_1^{d-1}} f(\Gamma' \varphi_g(K(v) D(T)))\, d\lambda(v) = \int_{\Gamma' \backslash \Gamma' H_g} f(x)\, d\mu_g(x)$$

The equidistribution is not necessarily in the whole n-dimensional space, but possibly in a smaller space that is still nice and homogeneous.

Theorem 3. $\Xi_T \to \Xi$

What is Ξ?

Let $\mathcal{L} = \delta^{1/n} \mathbb{Z}^n g$ for $g \in G'$ and $\delta > 0$. One can now show that for all $h \in H_g$, $\pi_{\mathrm{int}}(\delta^{1/n} \mathbb{Z} h g) \subset \mathcal{A}$ and, for almost every $h \in H_g$, $\overline{\pi_{\mathrm{int}}(\delta^{1/n} \mathbb{Z}^n h g)} = \mathcal{A}$. Thus we get a nice map $\Gamma' h \mapsto \mathcal{P}(W, \delta^{1/n} \mathbb{Z}^n h g) \subset \mathbb{R}^d$, and so, using the Siegel-Veech formula we have

$$\Xi = \mathcal{P}(W, \delta^{1/n} \mathbb{Z}^n h g).$$

$\Gamma' h$ distributed according to μ_g.

2.3. **Examples.**

Example 1. Let $\mathcal{L}$ be a lattice that sits generically in $\mathbb{R}^n$, then $H_g = \mathrm{SL}(n, \mathbb{R})$.

Example 2. The Penrose tiling has a nontrivial H_g, i.e. not just $\mathrm{SL}(n, \mathbb{R})$, but in fact is an embedded subgroup.

3. Hyperbolic Lattices

Let $\mathbb{H}^n$ be hyperbolic n-space, G the group of orientation preserving isometries, and Γ a lattice in G. (Note: you can ask the same questions when Γ is not necessarily a lattice, for example if Γ is geometrically finite).

The orbit $\bar{w} = \Gamma w$ for $w \in \mathbb{H}^n$, will be our point set. Define $\Gamma_w = \mathrm{Stab}_\Gamma(w)$. Then

$$\#(\bar{w} \cap B_t(z)) = \#(\gamma \in \Gamma / \Gamma_w \mid d(z, \gamma w) \leq t\} \sim c e^{(n-1)t}$$

We also know (from Nichols, '83) that points equidistribute when projected, that is

$$\frac{\#(\mathcal{P}_t \cap \mathcal{A})}{\mathcal{P}_t} \xrightarrow{t \to \infty} []\omega(\mathcal{A})$$

for all $\mathcal{A} \subset S_1^{d-1}$.

Consider the hyperbolic ball of radius t centered at i. We are interested in counting points in a cone of geodesic boundary in direction v. Applying the rotation $K(v)$, transforms the cone to a downward cone above a set of constant volume. To make this cone nicely proportioned, we want to apply the matrix $D(T)$, this raises the bottom of the cone to the center i and the top to the point ie^t. Letting t tend to infinity we get a limiting object which is a "cuspidal cone", where the top moves off to infinity.

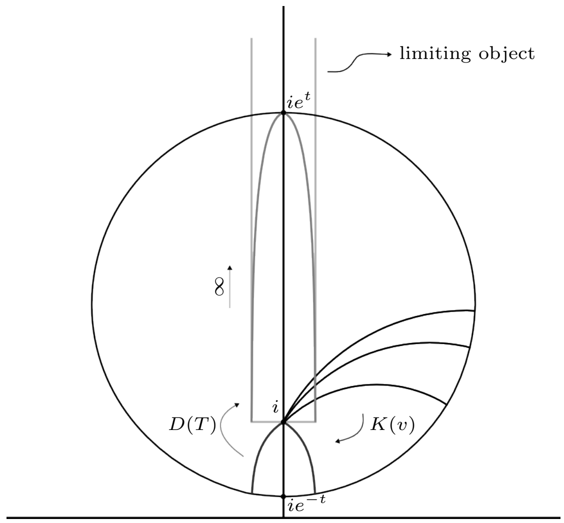

FIGURE 1. The limiting object in $\mathbb{H}^n$

Theorem 4 (Marklof-Vinogradov)**.** *In this setting we also have equidistribution of spherical averages for the limiting object shown in Fig. 1. Our limiting point process is given by a random hyperbolic lattice intersecting the cuspidal cone in k points.*